Homework 5 (Stat 620, Fall 2009)

Due Thu Oct 15, in class

1. Prove that if the number of state is n, and if state j is accessible from state i, then it is accessible in n or fewer steps.

   Solution: j is accessible from i if, for some $k \geq 0$, $P_{ij}^k > 0$. Now

   $$P_{ij}^k = \sum \prod_{m=1}^{k} P_{i_m i_{m+1}}$$

   where the sum is taken over all sequences $(i_0, i_1, \cdots, i_k) \in \{1, \cdots, n\}^{k+1}$ of states with $i_0 = i$ and $i_k = j$. Now, $P_{ij}^k > 0$ implies that at least one term is positive, say $\prod_{m=1}^{k} P_{i_m i_{m+1}} > 0$. If a state s occurs twice, say $i_a = i_b = s$ for $a < b$, and $(a, b) \neq (0, k)$, then the sequence of states $(i_0, \cdots, i_{a-1}, i_b, \cdots, i_k)$ also has positive probability, without this repetition. Thus, the sequence $i_0, \cdots, i_k$ can be reduced to another sequence, say $j_0, \cdots, j_r$, in which no state is repeated. This gives $r \leq n - 1$, so $i \neq j$ is accessible in at most $n - 1$ steps. If $i = j$, we cannot remove this repetition! This gives the possibility of $r = n$, when $i = j$, but there are no other repetitions.

2. For states i, j, k with $k \neq j$, let

   $$P_{ij/k}^n = P\{X_n = j, X_\ell \neq k, \ell = 1, ..., n-1 | X_0 = i\}.$$

   (a) Explain in words what $P_{ij/k}^n$ presents.

   (b) Prove that, for $i \neq j$, $P_{ij}^n = \sum_{k=0}^{n} P_{ii}^k P_{ij/i}^{n-k}$

   Solution:

   (a) $P_{ij/k}^n$ is the probability of being in j at time n, starting in i at time 0, while avoiding k.

   (b) Let N be the (random) time at which $\{X_k\}$ is last in i before time n. Then since $0 \leq N \leq n$,

   $$\begin{aligned} P_{ij}^n &= P[X_n = j | X_0 = i] \\ &= \sum_{k=0}^{n} P[X_n = j, N = k | X_0 = i] \\ &= \sum_{k=0}^{n} P[X_n = j, X_k = i, X_l \neq i : k+1 \leq l \leq n | X_0 = i] \\ &= \sum_{k=0}^{n} P[X_n = j, X_l \neq i : k+1 \leq l \leq n | X_0 = i, X_k = i] P[X_k = i | X_0 = i] \end{aligned}$$

   Now using the Markov property

   $$\begin{aligned} P_{ij}^n &= \sum_{k=0}^{n} P[X_n = j, X_l \neq i : k+1 \leq l \leq n | X_k = i] P[X_k = i | X_0 = i] \\ &= \sum_{k=0}^{n} P_{ij/i}^{n-k} P_{ii}^k \end{aligned}$$

Note that one can also calculate $P_{ij}^n = E[P[X_n = j|X_0 = i, N]]$, but this works out slightly less easily.

3. Show that the symmetric random walk is recurrent in two dimensions and transient in three dimensions.

   Comments: This asks you to extend the argument of Ross Example 4.2(A) to two and three dimensions. You may use either of the definitions of the simple symmetric random walk in d dimensions from the notes.

   <u>Solution</u>: Define the symmetric random walk in d dimensions, $X_n^{(d)} = (X_{(n,1)}, \cdots, X_{(n,d)})$, by

$$X_{n+1,j} = \begin{cases} X_{n,j} + 1 & \text{with probability } 0.5 \\ X_{n,j} - 1 & \text{with probability } 0.5 \end{cases}$$

   i.e. each component of $X_n^{(d)}$ carries out an independent symmetric random walk in one dimension. Let $P_n^{(d)} = P[X_n^{(d)} = X_0^{(d)}]$. From example 4.2(A), $P_{2n}^{(1)} \sim 1/\sqrt{\pi n}$, and $P_{2n+1}^{(1)} = 0$. By independence,

$$P_{2n}^{(d)} \sim (\frac{1}{\sqrt{\pi n}})^d$$

   Therefore,

$$\sum_{n=1}^{\infty} P_n^{(2)} \sim \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

   and

$$\sum_{n=1}^{\infty} P_n^{(3)} \sim \frac{1}{\pi^{3/2}} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < \infty$$

   Thus by Proposition 4.2.3 we get the recurrence of the two dimensional symmetric random walk, and the transience in three (or more) dimensions. Note that it would take a bit more work to fully justify our implicit "interchange" of $\sim$ with infinite summation.

4. A transition probability matrix P is said to be doubly stochastic if

$$\sum_i P_{ij} = 1 \quad \text{for all } j.$$

   That is, the column sums all equal 1. If a doubly stochastic chain has n states and is ergodic, calculate its limiting probabilities.

   Hint: guess the answer, and then show that your guess satisfies the required equations. Then, by arguing for the uniqueness of the limiting distribution, you will have solved the problem.

   <u>Solution</u>: Let $\mu = (1/n, \cdots, 1/n)$. Then since $\sum_j P_{ji} = 1$, $[\mu P]_i = \sum_{j=0}^{n-1} \mu_j P_{ji} = \sum_j P_{ji}/n = 1/n$. Thus, $\mu P = \mu$. The uniqueness of the limiting distribution for an ergodic Markov chain implies that any solution to $\mu P = \mu$ with $\mu_i > 0$ and $\sum_i \mu_i = 1$ is the required limiting distribution.

5. Jobs arrive at a processing center in accordance with a Poisson process with rate λ. However, the center has waiting space for only N jobs and so an arriving job finding N others waiting goes away. At most 1 job per day can be processed, and the processing of this job must

start at the beginning of the day. Thus, if there are any jobs waiting for processing at the beginning of a day, then one of them is processed that day, and if no jobs are waiting at the beginning of a day then no jobs are processed that day. Let X_n denote the number of jobs at the center at the beginning of day n.
(a) Find the transition probabilities of the Markov chain $\{X_n, n \geq 0\}$.
(b) Is this chain ergodic? Explain.
(c) Write the equations for the stationary probabilities.

Instructions:
(a). Suppose that the arrival rate λ has units day^{-1}.
(b). You may assert the property that a finite state, irreducible, aperiodic Markov chain is ergodic (see Theorem 4.3.3, the discussion following this theorem, and Problem 4.14).
(c). There is no particularly elegant way to write these equations, and you are not expected to solve them.

Solution: The state space is $0, 1, \cdots, N$. Let $p(j) = \lambda^j e^{-\lambda}/j!$
(a)

$$P_{0k} = \begin{cases} P[k \text{ arrivals}] = p(k) & \text{for } k = 0, \cdots, N-1 \\ P[\geq N \text{ arrivals}] = \sum_{l=N}^{\infty} p(l) & \text{for } k = N \end{cases}$$

$$P_{jk} = \begin{cases} P[k-j+1 \text{ arrivals}] = p(k-j+1) & \text{for } k = j-1, \cdots, N-1 \\ P[\geq N-j+1 \text{ arrivals}] = \sum_{l=N-j+1}^{\infty} p(l) & \text{for } k = N \end{cases}$$

(b) The Markov chain is irreducible, since $P_{0k} > 0$ and $P_{k0}^k = (p(0))^k > 0$. It is aperiodic, since $P_{00} > 0$. A finite state Markov chain which is irreducible and aperiodic is ergodic (since it is not possible for all states to be transient or for any states to be null recurrent).
(c) For $j < N$, the identity $\pi_j = \sum_k \pi_k P_{kj}$ becomes $\pi_j = \sum_{k=0}^{j+1} \pi_k P_{kj}$. This can be rewritten as a recursion

$$\pi_{j+1} = \frac{\pi_j(1 - P_{jj}) - \sum_{k=0}^{j-1} \pi_k P_{kj}}{P_{j+1,j}}$$

An alternative expression is given as follows. Since the long run rate of entering $\{0, \cdots, j\}$ must equal the rate of leaving $\{0, \cdots, j\}$

$$\pi_{j+1} p(0) = \pi_0 \overline{F}(j) + \sum_{k=1}^{j} \pi_k \overline{F}(j-k+1)$$

where $\overline{F}(j) = \sum_{k=j+1}^{\infty} p(k)$.

Recommended reading:
Sections 4.1 through 4.3, excluding examples 4.3(A,B,C).

Supplementary exercises: 4.13, 4.14
These are optional, but recommended. Do not turn in solutions—they are in the back of the book.