

## Homework 6 (Stat 620, Fall 2009)

Due Thu Nov 5, in class

1. In a branching process the number of offspring per individual has a Binomial  $(2, p)$  distribution. Starting with a single individual, calculate:
  - (a) the extinction probability;
  - (b) the probability that the population becomes extinct for the first time in the third generation.
  - (c) Suppose that, instead of starting with a single individual, the initial population size  $Z_0$  is a random variable that is Poisson distributed with mean  $\lambda$ . Show that, in this case, the extinction probability is given, for  $p > 1/2$ , by

$$\exp\{\lambda(1 - 2p)/p^2\}.$$

**Instructions:** (b) If  $X_n$  is the size of the  $n$ th generation, this question is asking you to find  $P(X_3 = 0, X_2 > 0, X_1 > 0 | X_0 = 1)$ . This can be done by brute force calculation, or by using probability generating functions.

2. Consider a time-reversible Markov chain with transition probabilities  $P_{ij}$  and limiting probabilities  $\pi_i$ ; and now consider the same chain truncated to the states  $0, 1, \dots, M$ . That is, for the truncated chain its transition probabilities  $\bar{P}_{ij}$  are

$$\bar{P}_{ij} = \begin{cases} P_{ij} + \sum_{k>M} P_{ik}, & 0 \leq i \leq M, j = i \\ P_{ij}, & 0 \leq i \neq j \leq M \\ 0, & \text{otherwise.} \end{cases}$$

Show that the truncated chain is also time reversible and has limiting probabilities given by

$$\bar{\pi}_i = \frac{\pi_i}{\sum_{i=0}^M \pi_i}$$

3. For an ergodic semi-Markov process:
  - (a) Compute the rate at which the process makes a transition from  $i$  into  $j$ .
  - (b) Show that  $\sum_i P_{ij}/\mu_{ii} = 1/\mu_{jj}$ .
  - (c) Show that the proportion of time that the process is in state  $i$  and headed for state  $j$  is  $P_{ij}\eta_{ij}/\mu_{ii}$  where  $\eta_{ij} = \int_0^\infty \bar{F}_{ij}(t) dt$ .
  - (d) Show that the proportion of time that the state is  $i$  and will next be  $j$  within a time  $x$  is

$$\frac{P_{ij}\eta_{ij}}{\mu_{ii}} F_{ij}^e(x),$$

where  $F_{ij}^e$  is the equilibrium distribution of  $F_{ij}$

**Hint:** all parts of this question can be done by defining appropriate renewal-reward processes. For (d), we use the definition  $F_{ij}^e(x) = \int_0^x \bar{F}_{ij}(y) dy / \int_0^\infty \bar{F}_{ij}(y) dy$  (see Ross, p131). This is the delay required to make a delayed renewal process with renewal distribution  $F_{ij}$  stationary. It arises here since it is also the limiting distribution of the residual life process for a non-lattice renewal process.

4. A taxi alternates between three locations. When it reaches location 1 it is equally likely to go next to either 2 or 3. When it reaches 2 it will next go to 1 with probability  $1/3$  and to 3 with probability  $2/3$ . From 3 it always goes to 1. The mean times between location  $i$  and  $j$  are  $t_{12} = 20, t_{13} = 30$  and  $t_{23} = 30$  (with  $t_{ij} = t_{ji}$ ).
- (a) What is the (limiting) probability that the taxi's most recent stop was at location  $i, i = 1, 2, 3$ ?
- (b) What is the (limiting) probability that the taxi is heading for location 2?
- (c) What fraction of time is the taxi traveling from 2 to 3? Note: Upon arrival at a location the taxi immediately departs.

**Recommended reading:**

Sections 4.5, 4.7, 4.8, 5.1, 5.2. You may skip Section 4.6, which will not be covered in this course.

**Supplementary exercise:** 4.40

These are optional, but recommended. Do not turn in solutions—they are in the back of the book.