Homework 7 (Stat 620, Fall 2009)

Due Thu Nov 12, in class

1. Show that a continuous-time Markov chain is regular, given (a) that $\nu_i < M < \infty$ for all i or (b) that the corresponding embedded discrete-time Markov chain with transition probabilities P_{ij} is irreducible and recurrent.

   Hint: For (a), you may follow the method suggested in the book solution (p. 491).

2. Let $P(t) = P_{00}(t)$.

   (a) Find $\lim_{t\to 0} \frac{1-P(t)}{t}$.

   (b) Show that $P(t)P(s) \le P(t+s) \le 1 - P(s) + P(s)P(t)$.

   (c) Show $|P(t) - P(s)| \le 1 - P(t-s)$, $s < t$ and conclude that P is continuous.

   Hint: For (c), one way to get these inequalities is to bound $P[X(t) = 0]$ by conditioning on $X(s)$, and to bound $P[X(t) \neq 0]$ by conditioning on $X(t-s)$.

3. Suppose that the "state" of the system can be modeled as a two-state continuous-time Markov chain with transition rates $\nu_0 = \lambda, \nu_1 = \mu$. When the state of the system is i, "events" occur in accordance with a Poisson process with rate α_i for $i = 0, 1$. Let $N(t)$ denote the number of events in $(0, t)$.

   (a) Find $\lim_{t\to\infty} N(t)/t$.

   (b) If the initial state is state 0, find $\mathbb{E}[N(t)]$.

   Hint For (a), one approach is to let return times into state 0 form a renewal process, and consider a reward to be the number of "events" in the renewal period. For (b), you may adopt the following informal approach. Supposing that one can write $E[N(t)] = \int_0^t E[dN(s)]$, you may then argue that $E[dN(s)|X(s) = i] = \alpha_i\, ds$, where $X(t)$ is the two-state Markov chain.

4. Consider a population in which each individual independently gives birth at an exponential rate λ and dies at an exponential rate μ. In addition, new members enter the population in accordance with a Poisson process with rate θ. Let $X(t)$ denote the population size at time t.

   (a) What type of process is $\{X(t), t \ge 0\}$?

   (b) What are its parameters?

   (c) Find $\mathbb{E}\big[X(T)|X(0) = i\big]$.

Recommended reading:
Sections 5.3, 5.4, 5.5.

Supplementary exercise: 5.14
Optional, but recommended. Do not turn in a solution—it is in the back of the book.