Some extra steps for applying the memoryless property in class # 4

$$\mathbb{P}\Big[X_{n+1}-(t-S_n)+\sum_{i=n+2}^{n+M} X_i \le s \;\Big|\; S_n \le t,\ X_{n+1} \ge t-S_n\Big]$$

$$= \frac{\mathbb{P}\Big[X_{n+1}-(t-S_n)+\sum_{i=n+2}^{n+M} X_i \le s,\ S_n \le t \;\Big|\; X_{n+1} \ge t-S_n\Big]}{\mathbb{P}[S_n \le t \mid X_{n+1} \ge t-S_n]}$$

$$= \frac{\int_0^t \mathbb{P}\Big[X_{n+1}-(t-S_n)+\sum_{i=n+2}^{n+M} X_i \le s \;\Big|\; S_n = u,\ X_{n+1} \ge t-S_n\Big]\, dF_{S_n \mid X_{n+1} \ge t-S_n}(u)}{\mathbb{P}[S_n \le t \mid X_{n+1} \ge t-S_n]}$$

$$= \frac{\int_0^t \mathbb{P}\Big[X_{n+1}-(t-u)+\sum_{i=n+2}^{n+M} X_i \le s \;\Big|\; S_n = u,\ X_{n+1} \ge t-u\Big]\, dF_{S_n \mid X_{n+1} \ge t-S_n}(u)}{\mathbb{P}[S_n \le t \mid X_{n+1} \ge t-S_n]}$$

$$= \frac{\int_0^t \mathbb{P}\Big[X_{n+1}-(t-u)+\sum_{i=n+2}^{n+M} X_i \le s \;\Big|\; X_{n+1} \ge t-u\Big]\, dF_{S_n \mid X_{n+1} \ge t-S_n}(u)}{\mathbb{P}[S_n \le t \mid X_{n+1} \ge t-S_n]}$$

Since $X_{n+1}, \dots X_{n+M}$ are independent of S_n

$$= \frac{\int_0^t \mathbb{P}\Big[X_{n+1}+\sum_{i=n+2}^{n+M} X_i \le s\Big]\, dF_{S_n \mid X_{n+1} \ge t-S_n}(u)}{\mathbb{P}[S_n \le t \mid X_{n+1} \ge t-S_n]}$$

$$= \frac{\mathbb{P}\Big[\sum_{i=n+1}^{n+M} X_i \le s\Big] \int_0^t dF_{S_n \mid X_{n+1} \ge t-S_n}(u)}{\mathbb{P}[S_n \le t \mid X_{n+1} \ge t-S_n]}$$

$$= \mathbb{P}\Big[\sum_{i=n}^{n+M} X_i \le s\Big]$$