3. Renewal Theory

- Definition 3 of the Poisson process can be generalized: Let $X_1, X_2, \ldots, \sim \text{iid}\, F(x)$ be non-negative **interarrival times**. Set $S_n = \sum_{i=1}^{n} X_i$ and $N(t) = \max\{n : S_n \leq t\}$. Then $\{N(t)\}$ is a **renewal process**.

- Set $\mu = \mathbb{E}[X_n]$. We assume $\mu > 0$ (i.e., $F(0)<1$) but we allow the possibility $\mu = \infty$.

- Many questions about more complex processes can be addressed by identifying a relevant renewal process. Therefore, we study renewal theory in some detail.

- The **renewal function** is $m(t) = \mathbb{E}[N(t)]$.

- Writing the c.d.f. of S_n as $F_n = F * F * \ldots * F$, the **n-fold convolution** of F, we have

$$\begin{aligned}
&\mathbb{P}[N(t) = n] \\
&\quad = \mathbb{P}[S_n \leq t] - \mathbb{P}[S_{n+1} \leq t] \\
&\quad = F_n(t) - F_{n+1}(t).
\end{aligned}$$

Recall: if X and Y are independent, with c.d.f. F_X and F_Y respectively, then

$$F_{X+Y}(z) = \int_{-\infty}^{\infty} F_X(z-y)dF_Y(y) = \int_{-\infty}^{\infty} F_Y(z-x)dF_X(x).$$

This identity is called a **convolution**, written

$$F_{X+Y} = F_X * F_Y = F_Y * F_X.$$

Taking Laplace transforms gives $\tilde{F}_{X+Y} = \tilde{F}_X \tilde{F}_Y$.

- It follows that

$$\begin{aligned} m(t) &= \mathbb{E}[N(t)] = \sum_{n=1}^{\infty} n\big(F_n(t) - F_{n+1}(t)\big) \\ &= \sum_{n=1}^{\infty} F_n(t) \end{aligned}$$

Limit Properties for Renewal Processes

(L1) $\boxed{\lim_{n\to\infty} S_n/n = \mu}$ with probability 1.

Why?

(L2) $\boxed{\lim_{t\to\infty} N(t) = \infty}$ with probability 1.

Why?

(L3) $\boxed{\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}}$ with probability 1.

Why?

(L4) **The Elementary Renewal Theorem**

$$\boxed{\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}}$$

- Why is L4 different from L3? Because it is not always true that, for any stochastic process, $\mathbb{E}\left[\lim_{t\to\infty} \frac{N(t)}{t}\right] = \lim_{t\to\infty} \mathbb{E}\left[\frac{N(t)}{t}\right]$.

<u>counter-example</u>

Exchanging Expectation and Summation

- If X_1, $X_2, \ldots$ are non-negative random variables, $\mathbb{E}\left[\sum_{n=1}^{\infty} X_n\right] = \sum_{n=1}^{\infty} \mathbb{E}[X_n]$.
- For any X_1, $X_2, \ldots$ with $\sum_{n=1}^{\infty} \mathbb{E}\big[|X_n|\big] < \infty$, $\mathbb{E}[\sum_{n=1}^{\infty} X_n] = \sum_{n=1}^{\infty} \mathbb{E}[X_n]$.
- These results do not require independence; they are consequences of two basic theorems. (checking this is an optional exercise.)

Monotone Convergence Theorem: If $Y_1, Y_2, \ldots$ is an increasing sequence of random variables (i.e., $Y_n \leq Y_{n+1}$) then $\mathbb{E}\big[\lim_{n\to\infty} Y_n\big] = \lim_{n\to\infty} \mathbb{E}[Y_n]$.

Dominated Convergence Theorem: For any $Y_1, Y_2, \ldots$, if there is a random variable Z with $\mathbb{E}[Z] < \infty$ and $|Y_n| < Z$ for all n, then $\mathbb{E}\big[\lim_{n\to\infty} Y_n\big] = \lim_{n\to\infty} \mathbb{E}[Y_n]$.

Stopping Times

- A non-negative integer-valued random variable N is a **stopping time** for a sequence $X_1, X_2, \ldots$ if $\mathbb{E}[N] < \infty$ and $\{N = n\}$ is determined by $X_1, \ldots, X_n$.
- **Example**: If cards are dealt successively from a deck, the position of the first spade is a stopping time, say N. Note that $N - 1$, the position of the last card before a spade appears, is **not** a stopping time (but $N + 1$ is).
- **Example**: Any gambling strategy cannot depend on future events! Thus, the decision to stop gambling must be a stopping time.
- If $X_1, X_2, \ldots$, are independent, then $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \ldots$ if N is a stopping time. (This is the definition in Ross, p. 104, but we will use the more general definition, which will be needed later).

Example For a renewal process, $N(t) + 1$ is a stopping time for the interarrival sequence $X_1, X_2, \ldots$

(Is $N(t)$ a stopping time? Why.)

(i) Show $\{N(t) + 1 = n\}$ is determined by $X_1, \ldots, X_n$

(ii) Show that $E[N(t)] < \infty$, i.e., check that

$$\boxed{m(t) < \infty}$$

Wald's Equation: If $X_1, X_2, \ldots$ are iid with $\mathbb{E}[X_1] < \infty$ and N is a stopping time, then

$$\boxed{\mathbb{E}\left[\sum_{n=1}^{N} X_n\right] = \mathbb{E}[N]\,\mathbb{E}[X_1]}$$

Proof

Proof of the elementary Renewal Theorem

(i) Apply Wald's equation to the stopping time $N(t) + 1$, to get $\liminf_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}$

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note that we use the definition

$\liminf_{t\to\infty} f(t) = \lim_{t\to\infty} \{\inf_{s>t} f(s)\}$

(ii) Show $\limsup_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu}$ by applying Wald's equation to a **truncated** renewal process, with

$$\widetilde{X}_n = \begin{cases} X_n & \text{if } X_n \leq M \\ M & \text{if } X_n > M \end{cases}$$

and $\widetilde{S}_n = \sum_{i=1}^n \widetilde{X}_i$, $\widetilde{N}(t) = \sup\left\{n : \widetilde{S}_n \leq t\right\}$, $\widetilde{\mu} = \mathbb{E}\left[\widetilde{X}\right]$, $\widetilde{m}(t) = \mathbb{E}\left[\widetilde{N}(t)\right]$.

Example. Customers arrive at a bank as a Poisson process, rate λ. There is only one clerk, and service times are iid with distribution G. Customers only enter the bank if the clerk is available.

Note: This is an $M/G/1$ queue with no buffer.

(a) Identify a relevant renewal process.

(b) At what rate do customers enter the bank? (what does "rate" mean here?)

(c) What fraction of potential customers enter the bank?

(d) What fraction of time is the server busy?

(L5) **Central Limit Theorem for $N(t)$.**

Suppose $\mu = \mathbb{E}[X_1] < \infty$ and $\sigma^2 = \mathrm{Var}(X_1) < \infty$. Then,

$$\lim_{t\to\infty} \mathbb{P}\left[\frac{N(t)-t/\mu}{\sigma\sqrt{t/\mu^3}} < y\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2}\, dx$$

- This is a consequence of the central limit theorem for S_n (see Ross, Theorem 3.3.5 for a proof).
- Why is there a μ^3 in the denominator? We can give a dimension analysis, as a check:

- A diagram and an informal argument can suggest a method of proof, and help explain the result:

(L6) **The Key Renewal Theorem**

$$\lim_{t\to\infty}\int_0^t h(t-x)\,dm(x) = \frac{1}{\mu}\int_0^\infty h(t)\,dt$$

Under the requirements that

(i) F is not **lattice**, i.e. there is no $d \geq 0$ such that $\sum_{n=0}^{\infty}\mathbb{P}\big[X_1 = nd\big] = 1$.

(ii) $h(t)$ is directly Riemann integrable (see following slide).

- L6 can be seen as a result on sums over a moving window: suppose at time t we consider a sum H_t with contribution $h(t-x)$ for an event at time $t-x$. Then,

$$H_t = \sum_{n=0}^{N(t)-1} h(t - S_{N(t)-n}) = \int_0^t h(t-x)dN(x).$$

Exchanging integration and expectation, we get

$$\mathbb{E}[H_t] = \int_0^t h(t-x)\,\mathbb{E}[dN(x)] = \int_0^t h(t-x)\,dm(x).$$

Example: Set $h(t) = \begin{cases} 1 & \text{for } t \leq a \\ 0 & \text{else} \end{cases}$.

Then H_t counts the number of events in the time window $[t-a, t]$. In this case, since $h(t)$ is directly Riemann integrable, (L6) gives the following result:

(L7) **Blackwell's theorem, part (i)**

If F is not lattice, then, for $a > 0$,

$$\boxed{\lim_{t\to\infty} m(t) - m(t-a) = a/\mu}$$

How does this follow?

- If F is lattice, then they key renewal theorem fails: **counter-example**...

- A modification of L(7)(i) still holds:

(L7) **Blackwell's theorem, part (ii)**

If F is lattice with period d then

$$\boxed{\lim_{n\to\infty} \mathbb{E}\big[\text{number of renewals at } nd\big] = \frac{d}{\mu}}$$

- Note that the number of renewals at nd can be written as $dN(nd)$.

<u>Definition</u>: $h : [0, \infty] \to \mathbb{R}$ is **directly Riemann integrable** (dRi) if

$$\lim_{a \to 0} a \sum_{n=1}^{\infty} \sup_{(n-1)a \le t \le na} h(t) = \lim_{a \to 0} \sum_{n=1}^{\infty} \inf_{(n-1)a \le t \le na} h(t).$$

- i.e., if the limits of upper and lower bounds exist and are equal, then h is dRi and the integral is equal to this limit.

- $h : [0, T] \to \mathbb{R}$ is **Riemann integrable** (Ri) if

$$\lim_{a \to 0} a \sum_{n=1}^{T/a} \sup_{(n-1)a \le t \le na} h(t) = \lim_{a \to 0} \sum_{n=1}^{T/a} \inf_{(n-1)a \le t \le na} h(t).$$

  and $h : [0, \infty] \to \mathbb{R}$ is **Riemann integrable** if the limit $\int_0^\infty h(t)\, dt = \lim_{T \to \infty} \int_0^T h(t)\, dt$ exists.

- The dRi definition requires the ability to take a limit (i.e., an infinite sum) inside the $\lim_{a \to 0}$. This infinite sum does not always exist for a Riemann integrable function.

- It is possible to find an $h(t)$ which is Ri but not dRi and for which the key renewal theorem fails. (e.g., Feller, "An Introduction to Probability Theory and its Applications, Vol. II").
- If $h(t)$ is dRi then $h(t)$ is Ri, and the two integrals are equal.
- If $h(t)$ is Ri and $h(t) \geq 0$ and $h(t)$ is non-increasing then it can be shown that $h(t)$ is dRi (this is sometimes an easy way to show dRi).

Example (Alternating Renewal Process)

- Each interarrival time X_n consists of an "on-time" Z_n followed by an "off-time" Y_n. Suppose (Y_n, Z_n) are iid, but Y_n and Z_n may be dependent.

- Suppose Y_n and Z_n have marginal distributions G and H and $X_n = Y_n + Z_n \sim F$.

- E.g., let $Z_n = \begin{cases} X_n & \text{if } X_n \le x \\ x & \text{if } X_n > x \end{cases}$.

  Then the alternating process is "on" if $t - S_{N(t)} \le x$. The quantity $A(t) = t - S_{N(t)}$ is the **age** of the renewal process at time t.

- E.g., let $Y_n = \begin{cases} X_n & \text{if } X_n \le x \\ x & \text{if } X_n > x \end{cases}$.

  In this case, the process is "off" if $S_{N(t)+1} - t \le x$. This alternating process is appropriate to study the **residual life** $Y(t) = S_{N(t)+1} - t$.

(L8) **Alternating Renewal Theorem**

If $E[Z_1 + Y_1] < \infty$ and F is non-lattice, then

$$\lim_{t \to \infty} \mathbb{P}\big[\text{system is "on" at } t\big] = \frac{\mathbb{E}[Z_1]}{\mathbb{E}[Z_1] + E[Y_1]}$$

Proof. The method is to condition on $S_{N(t)}$ and apply the key renewal theorem. This is an example of a powerful conditioning approach.

Proof continued

Example (Regenerative Process)

- A stochastic process $\{X(t), t \geq 0\}$ is **regenerative** if there are random times $S_1, S_2, \ldots$ at which the process **restarts**. This means that $\{X(t+u), t \geq 0\}$ given $S_n = u$ is conditionally independent of $\{X(t), 0 \leq t \leq u\}$ and has the same distibution as $\{X(t), t \geq 0\}$.

- A regenerative process defines a renewal process with $X_n = S_n - S_{n-1}$ and $N(t) = \max\{n : S_n \leq t\}$. Each interarrival interval for $\{N(t)\}$ is called a **cycle** of $\{X(t)\}$.

- An alternating renewal process is a regenerative process:

  An alternating renewal process $X(t)$ takes values "on" and "off", and the times at which $X(t)$ switches from "off" to "on" are renewal times, by the construction of the alternating renewal process.

(L9) **Regenerative Process Theorem**

If $X(t)$ takes values $0, 1, 2, \ldots$ and the expected length of a cycle is finite then

$$\lim_{t \to \infty} \mathbb{P}[X(t) = j] = \frac{\mathbb{E}[\text{time in state } j \text{ for one cycle}]}{\mathbb{E}[\text{time of a cycle}]}$$

Outline of Proof

- **Example**: If $A(t)$ and $Y(t)$ are the age and residual life processes of a renewal process $N(t)$, how can (L9) be applied to find $\lim_{t\to\infty} \mathbb{P}\left[Y(t) > x, A(t) > y\right]$?

Renewal Reward Processes

- At each renewal time S_n for a renewal process $\{N(t)\}$ we receive a **reward** R_n. We think of the reward as accumulating over the time interval $[S_{n-1}, S_n]$, so we assume that R_n depends on X_n but is independent of (R_m, X_m) for $m \neq n$. In other words, (R_n, X_n) is an iid sequence of vectors.
- The **reward process** is $R(t) = \sum_{n=1}^{N(t)} R_n$.

(L10) If $\mathbb{E}[R_1] < \infty$ and $\mathbb{E}[X_1] < \infty$ then

(i) $\boxed{\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R_1]}{\mathbb{E}[X_1]}}$ with probability 1

(ii) $\boxed{\lim_{t\to\infty} \frac{\mathbb{E}[R(t)]}{t} = \frac{\mathbb{E}[R_1]}{\mathbb{E}[X_1]}}$

Outline of proof

Example: A component is replaced if it reaches the age T years, at a "planned replacement" cost C_1. If it fails earlier, it is replaced at a "failure replacement" cost $C_2 > C_1$. Components have iid lifetimes, with distribution F. Find the long run cost per unit time.

Solution

Delayed Renewal Processes

- All the limit results for renewal processes still apply when the first arrival time has a different distribution from subsequent interarrival times.
- This situation arises often because the time $t = 0$ is the special time at which we start observing a process.

Example: Suppose cars on the freeway pass an observer with independent inter-arrival times, i.e., as a renewal process. Unless the observer starts counting when a car has just passed, the time to the first arrival will have a different distribution to subsequent intarrivals [an exception to this is when the interarrival times are memoryless!]

- Formally, let $X_1, X_2, \ldots$ be independent, with $X_1 \sim G$, $X_n \sim F$ for $n \geq 2$, and $S_n = \sum_{i=1}^{n} X_i$. Then $N_D(t) = \sup\{n : S_n \leq t\}$ is a **delayed renewal** process

- Defining $m_D(t) = \mathbb{E}[N_D(t)]$ and $\mu = \mathbb{E}[X_2]$,

(LD1) $\lim_{n\to\infty} \frac{S_n}{n} = \mu$ w.p.1

(LD2) $\lim_{t\to\infty} N_D(t) = \infty$ w.p.1

(LD3) $\lim_{t\to\infty} \frac{N_D(t)}{t} = \frac{1}{\mu}$ w.p.1

(LD4) $\lim_{t\to\infty} \frac{m_D(t)}{t} = \frac{1}{\mu}$

(LD5)

$$\lim_{t\to\infty} \mathbb{P}\left[\frac{N_D(t) - t/\mu}{\sigma\sqrt{t/\mu^3}} < y\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2}\,dx$$

(LD6) If F is not lattice and h is directly Riemann integrable,

$\lim_{t\to\infty} \int_0^t h(t-x)\,dm_D(x) = \frac{1}{\mu}\int_0^\infty h(t)dt$

(LD7) (i) If F is not lattice,

$\lim_{t\to\infty} m_D(t+a) - m_D(t) = a/\mu$

(ii) If F and G are lattice, period d,

$\lim_{n\to\infty} E\big[\text{number of renewals at } nd\big] = d/\mu$

Example: Suppose a coin lands H with probability p. The coin is tossed until k consecutive heads occur in a row. Show that the expected number of tosses is $\sum_{i=1}^{k}(1/p)^i$.

Solution

Solution continued