Applied Probability Qualifying Review Exam, May 2007
Solutions

1. (i)

$$P[Y_1(t) = y_1, Y_2(t) = y_2] = E[P[Y_1(t) = y_1, Y_2(t) = y_2 | Y_1(t) + Y_2(t) = y_1 + y_2]].$$

Using the properties that $\min(Z_{ij}, Z_{zj}) \sim$ Exponential $(\lambda_1 + \lambda_2)$, and $\{Z_{ij} < Z_{2j}\}$ is independent of min (Z_{ij}, Z_{ij}) with $P[Z_{ij} < Z_{ij}] = \frac{\lambda_1}{\lambda_1+\lambda_2}$, we have

$$P[Y_1(t) = y_1, Y_2(t) = y_2 | Y_1(t) + Y_2(t) = y_1 + y_2] = \binom{y_1 + y_2}{y_1} \frac{\lambda_1^{y_1} \lambda_2^{y_2}}{(\lambda_1 + \lambda_2)^{y_1+y_2}}$$

and so

$$\begin{aligned}
&P[y_1(t) = y_1, y_2(t) = y_2] \\
&= \binom{N}{y_1 + y_2} (1 - e^{-(\lambda_1+\lambda_2)t})^{y_1+y_2} (e^{-(\lambda_1+\lambda_2)t})^{N-y_1-y_2} \binom{y_1 + y_2}{y_1} \frac{\lambda_1^{y_1} \lambda_2^{y_2}}{(\lambda_1 + \lambda_2)^{y_1+y_2}} \\
&= \binom{N}{y_1 \;\; y_2 \;\; (N - y_1 - y_2)} p_1^{y_1} p_2^{y_2} (1 - p_1 - p_2)^{N-y_1-y_2}
\end{aligned}$$

for $p_i = [\lambda_i/(\lambda_1 + \lambda_2)](1 - \exp\{-(\lambda_1 + \lambda_2)t\})$.

1. (ii) If the n^{th} death event occurs at time τ_n, there are $N - n$ individuals remaining at τ_n. Due to the memoryless property of the Exponential distribution, the distribution of $Z_{ij} - \tau_n | Z_{ij} > \tau_n$ is Exponential (λ_i). Thus, the time to the next death event has distribution

$$\tau_{n+1} - \tau_n \sim \min(Z'_{1,1}, \ldots, Z'_{1,N-n}, Z'_{2,1}, \ldots, Z'_{2,N-n})$$

where Z'_{ij} are independent Exponential (λ_i) random variables. $\tau_{n+1} - \tau_n$ therefore has Exponential $((N - n)(\lambda_1 + \lambda_2))$ distribution, and the corresponding event has probability $\frac{\lambda_i}{\lambda_1+\lambda_2}$ of being type i. Thus

$$\begin{aligned}
\nu_{(y_1,y_2)} &= (N - n)(\lambda_1 + \lambda_2) \\
P_{(y_1,y_2)(y_1+1,y_2)} &= \frac{\lambda_1}{\lambda_1 + \lambda_2} \\
P_{(y_1,y_2)(y_1,y_2+1)} &= \frac{\lambda_2}{\lambda_1 + \lambda_2}
\end{aligned}$$

it also follows that

$$\begin{aligned}
\nu_{(y_1,y_2)(y_1+1,y_2)} &= \lambda_1(N - n) \\
\nu_{(y_1,y_2)(y_1,y_2+1)} &= \lambda_2(N - n)
\end{aligned}$$

2. (i)

$$\begin{aligned} P[M(s+t)-M(s)=k] &= E[P[M(s+t)-M(s)=k|G(s+t)-G(s)]] \\ &= \int \frac{(\lambda y)^k e^{-\lambda y}}{k!} \frac{y^{at-1}e^{-y/b}}{b^{at}\Gamma(at)}\, dy. \qquad (1) \end{aligned}$$

This depends on t but not on s, which is the stationary increment property for $M(t)$.

$$\begin{aligned} &P[M(t_k)-M(t_{k-1})=m_k, k=1,\ldots,n] \\ &= E[P[(N(G(t_k)) = N(G(t_{k-1})) = M_k, k=1,\ldots,n|\{G(t)\}]] \\ &= E[\prod_{k=1}^{n} P[N(G(t_k)) - N(G(t_{k-1})) = M_k|G(t_k)-G(t_{k-1})]] \qquad (2) \\ &= \prod_{k=1}^{n} EP[N(G(t_k)) - N(G(t_{k-1})) = m_k|G(t_k)-G(t_{k-1})] \qquad (3) \\ &= \Pi_{k=1}^{n} P[M(t_k)-M(t_{k-1}) = M_k]. \qquad (4) \end{aligned}$$

(2) follows from the independent increment property of $N(t)$. (3) follows from the independent increment property of $G(t)$. (4) is the independent increment property of for $M(t)$.

2. (ii)

$$\begin{aligned} E[M(s+t)-M(s)] &= E[E[M(s+t)-M(s)|\{G(t)\}]] \\ &= E[\lambda\{G(s+t)-G(s)\}] \\ &= \lambda abt \\ Var[M(s+t)-M(s)] &= E[Var[M(s+t)-M(s)|\{G(t)\}]] \\ &+ Var[E[M(s+t)-M(s)|\{G(t)\}]] \\ &= E[\lambda\{G(s+t)-G(s)\}] + Var[\lambda(G(s+t)-G(s))] \\ &= \lambda abt + \lambda^2 ab^2 t \end{aligned}$$

2. (iii) From (1),

$$\begin{aligned} P[M(s+t)-M(s)=k] &= \frac{\Gamma(k+at)\,\lambda^k\,(\frac{b}{\lambda b+1})^{k+at}}{\Gamma(at)\,k!\,b^{at}} \int \frac{y^{k+at-1}e^{-y\left(\frac{\lambda b+1}{b}\right)}}{(\frac{b}{\lambda b+1})^{k+at}\Gamma(k+at)}\, dy \\ &= \frac{\Gamma(k+at)(\lambda b)^k}{\Gamma(at)\,k!\,(\lambda b+1)^{k+at}} \end{aligned}$$

2. (iv) For a Poisson process, the mean and variance of increments are equal - here, they are not.

2. (v) A counting process with stationary independent increments for which $P[M(s+t) - M(s) \geq 2] = o(t)$ is a Poisson process. Thus, from (iv),

$$\lim_{t \to 0} \frac{P[M(s+t) - M(s) \geq 2]}{t} > 0,$$

assuming that this limit exists.

3. (i) Let $Z(n)$ be a random walk on Z with $Z(0) = 0$, $P[Z(n+1) = Z(n) + 1] = p$, and $P[Z(n+1) = Z(n) - 1] = 1 - p$. Notice that

$$P[\text{first loop is a clockwise loop}] = P_c = P[\ Z(n) \text{ hits } N \text{ before } -N].$$

Now, find $0 < \alpha < 1$ such that $U(n) = \alpha^{Z(n)}$ is a martingale, i.e.,

$$p\alpha + (1-p)\alpha^{-1} = 1.$$

(the existence of such an alpha was demonstrated in class). Then apply the martingale stopping thereom with stopping time $T = \inf\{n : Z(n) = N \text{ or } Z(n) = -N\}$. The stopping theorem holds since $U(\min(n, T))$ is bounded. This gives

$$E[\alpha^{Z(T)}] = E[\alpha^{Z(0)}] = 1$$

and so

$$\begin{aligned} P_c\alpha^N + (1 - P_c)\alpha^{-N} &= 1 \\ P_c &= \frac{1 - \alpha^{-N}}{\alpha^N - \alpha^{-N}} \end{aligned}$$

3. (ii) Set $V(n) = Z(n) - n(2p - 1)$, and again apply the martingale stopping theorem for T (which holds since $V(n+1) - V(n)$ is bounded and $E[T] < \infty$). This gives $E[V(T)] = E[V(0)] = 0$ and so

$$\begin{aligned} NP_c + (-N)(1 - P_c) - (2p-1)E[T] &= 0 \\ E[T] &= N(2P_c - 1)/(2p - 1). \end{aligned}$$

Now, let $R(n)$ be a renewal reward process, setting renewal times to be loop times and awarding a reward of 1 if the loop time is a clockwise loop time. Then,

$$\begin{aligned} \lim_{n \to \infty} \frac{E[R(n)]}{n} &= \frac{E[\text{reward on 1 cycle}]}{E[\text{time of 1 cycle}]} \\ &= \frac{P_c}{E[T]} = \frac{P_c(2p-1)}{N(2P_c - 1)} \end{aligned}$$

4. For $z > \delta > 0$, a symmetry argument (see notes, or Ross Section 8.2) gives

$$P[\max_{0 \leq t \leq 1} W(t) \geq z, W(1) \leq \delta] \ = \ P[W(1) \leq 2z - \delta] = \Phi(\delta - 2z),$$

where $\Phi(.)$ and $\phi(.)$ are the standard normal c.d.f. and p.d.f. respectively. Therefore,

$$\begin{aligned} P[\max_{0\leq t\leq 1} W(t) \geq z,\ 0 \leq W(1) \leq \delta] &= \Phi(\delta - 2z) - \Phi(-2z) \\ &= \delta\phi(-2z) + O(\delta) \\ &= \delta\phi(2z) + O(\delta) \end{aligned}$$

Now,

$$\begin{aligned} P[Z \geq z] &= P[\max_{0\leq t\leq 1} W(t) \geq z \mid W(1) = 0] \\ &= \lim_{\delta\to 0} P[\max_{0\leq t\leq 1} W(t) \geq z \mid 0 \leq W(1) \leq \delta] \\ &= \lim_{\delta\to 0} P[\max_{0\leq t\leq 1} W(t) \geq z, 0 \leq W(1) \leq \delta]/P[0 \leq W(1) \leq \delta] \\ &= \delta \overset{lim}{\rightarrow} 0 \ \frac{\delta\phi(2z) + 0(\delta)}{\delta\phi(0) + O(\delta)} = \frac{\phi(2z)}{\phi(0)} \\ &= (\frac{1}{\sqrt{2\pi}} e^{-(2z2/2)})/(\frac{1}{\sqrt{2\pi}}) = e^{-2z^2} \end{aligned}$$

So the density of Z is $f_Z(z) = 4ze^{-2z^2}$ for $z \geq 0$.