Qualifying Review Exam in Probability, Part I (Applied)

1. A population of N individuals dies of one of two causes, according to a stochastic process constructed as follows. At $t = 0$, a two dimensional array of independent, exponentially distributed random variables Z_{ij}, $i \in \{1, 2\}$, $j \in \{1, \dots, N\}$ is generated, with $E[Z_{ij}] = 1/\lambda_i$. Individual j dies at time Z_{ij} of cause i if $Z_{ij} = \min(Z_{1j}, Z_{2j})$. Thus, each individual has two potential death times for each cause and the individual dies from the first potential cause which arises. Let $Y_i(t)$ count the individuals deceased from cause i in the time interval $[0, t]$, and let $X(t) = (Y_1(t), Y_2(t))$.

   (i) [5 points] Find the probability mass function of the marginal distribution of $X(t)$ at a fixed time t, i.e., find $P[Y_1(t) = y_1, Y_2(t) = y_2]$.

   (ii) [5 points] Argue that $X(t)$ is a Markov chain, and find the transition rates and transition probabilities.

   Hint: one approach for (ii) is to show how a process with the same distribution as $X(t)$ has a standard construction of a Markov chain, namely

   (a) after entering state i, the amount of time it spends in the state before making a transition to a different state is exponentially distributed with rate, say ν_i.

   (b) when the process leaves state i, it will next enter state j with some probability P_{ij}.

2. Let $N(t)$ be a Poisson counting process, rate λ. Let $G(t)$ be a Gamma process, with parameters a and b, which is defined as follows. $G(t)$ has stationary independent increments, with $G(0) = 0$ and $G(s+t) - G(s) \sim Gamma(at, b)$, where $Y \sim Gamma\ (\alpha, \beta)$ is a non-negative random variable with density $f_Y(y) = y^{\alpha-1}e^{-y/\beta}\big/\Gamma(\alpha)\beta^{\alpha}$. For the present purposes, all we need to know about the function $\Gamma(\alpha)$ is that it is defined to make $f_Y(y)$ integrate to one. We will also need to know that the first two moments of Y are $E[Y] = \alpha\beta$ and $Var(Y) = \alpha\beta^2$.

   Define a counting process $M(t)$ by $M(t) = N(G(t))$.

   (i) [3 points] Show that $M(t)$ has independent, stationary increments.

   (ii) [3 points] Find $E[M(s+t) - M(s)]$ and $Var[M(s+t) - M(s)]$.

   (iii) [2 points] Show that

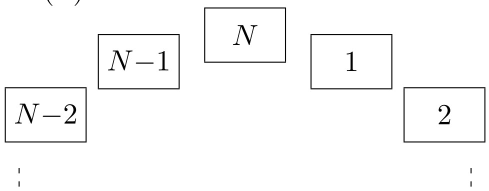
   $$P[M(s+t) - M(s) = k] = \frac{\Gamma(k + at)(\lambda b)^k}{\Gamma(at)\, k!\, (\lambda b + 1)^{k+at}}.$$

   (iv) [1 point] Explain why it follows from (ii) that $M(t)$ is not a Poisson Process, even though it is a counting process with stationary independent increments.

   (v) [1 point] What can you deduce via (iv) about

   $$\lim_{t \to 0} \frac{P[M(s+t) - M(s) \geq 2]}{t}.$$

3. Let $X(n)$ be a random walk on the integers $1, \ldots, N$ arranged clockwise in a circle,



Suppose $X(0) = N$ and

$$P[X(n+1) = X(n) + 1 \text{ modulo } N] = p \qquad \text{“clockwise step”}$$
$$P[X(n+1) = X(n) - 1 \text{ modulo } N] = (1-p) \qquad \text{“anticlockwise step”}$$

with $1/2 < p < 1$. Say that $X(n)$ completes a clockwise loop when $X(n)$ re-enters state N from state $N-1$, having last exited N into state 1. Formally, if $0 = T_0 \leq T_1 < T_2 < \ldots$ are successive times at which $X(T_n) = N$, then $X(n)$ completes a clockwise loop at time T_n if $X(T_{n-1} + 1) = 1$ and $X(T_n - 1) = N - 1$. Anticlockwise loops are defined correspondingly.

(i) [5 points] Show that

$$P[\text{the first clockwise loop occurs before an anticlockwise loop}] = P_c = \frac{1 - \alpha^{-N}}{\alpha^N - \alpha^{-N}}$$

where α solves $p\alpha + (1-p)\alpha^{-1} = 1$.

(ii) [5 points] Let $C(n)$ count the number of clockwise loops completed by time n. Find

$$\lim_{n \to \infty} \frac{E[C(n)]}{n}.$$

You may write your expression in terms of P_c and α.

4. Let $B(t)$ be a Brownian Bridge, i.e., $B(t)$ has the same distribution as a standard Brownian motion $W(t)$ conditioned on $W(1) = W(0) = 0$. Let

$$Z = \max_{0 \leq t \leq 1} B(t).$$

Find the probability density function of Z.

Hint: One approach involves finding an expression for

$$P[\max_{0 \leq t \leq 1} W(t) \geq Z,\ 0 \leq W(1) \leq \delta],$$

followed by a conditioning argument and taking an appropriate limit.