Sample questions, Exam I

1. Suppose that $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent Poisson processes with rates λ_1 and λ_2. Show that $\{N_1(t) + N_2(t), t \geq 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$. Also, show that the probability that the first event of the combined process comes from $\{N_1(t), t \geq 0\}$ is $\lambda_1/(\lambda_1 + \lambda_2)$, independently of the time of the event.

2. Suppose that shocks occur according to a Poisson process with rate λ, and suppose that each shock, independently, causes the system to fail with probability p. Let N denote the number of shocks that it takes for the system to fail and let T denote the time of failure. Find $P\{N = n | T = t\}$.

3. Consider an elevator that starts in the basement and travels upward. Let N_i denote the number of people that get in the elevator at floor i. Assume the N_i are independent and that N_i is Poisson with mean λ_i. Each person entering at i will, independent of everything else, get off at j with probability P_{ij}. $\sum_{j>i} P_{ij} = 1$. Let O_j = number of people getting off the elevator at floor j.

   (a) Compute $E[O_j]$.

   (b) What is the distribution of O_j?

   (c) What is the joint distribution of O_j and O_k?

4. If F is the uniform (0, 1) distribution function show that the renewal function is

$$m(t) = e^t - 1, \qquad 0 \leq t \leq 1.$$

   Now argue that the expected number of uniform (0, 1) random variables that need to be added until their sum exceeds 1 has mean e.

5. Packages arrive at a mailing depot in accordance with a Poisson process having rate λ. Trucks, picking up all waiting packages, arrive in accordance to a renewal process with nonlattice interarrival distribution F. Let $X(t)$ denote the number of packages waiting to be picked up at time t.

   (a) What type of stochastic process is $\{X(t), t \geq 0\}$?

   (b) Find an expression for

$$\lim_{t \to \infty} P\{X(t) = i\}, \qquad i \geq 0.$$

6. At the beginning of every time period, each of N individuals is in one of three possible conditions: infectious, infected but not infectious, or noninfected. If a noninfected individual becomes infected during a time period then he or she will be in an infectious condition during the following time period, and from then on will be in an infected (but not infectious) condition. During every time period each of the $\binom{N}{2}$ pairs of individuals are independently in contact with probability p. If a pair is in contact and one of the members of the pair is infectious and the other is noninfected then the noninfected person becomes infected (and is thus in the infectious condition at the beginning of the next period). Let X_n and Y_n denote the number of infectious and the number of noninfected individuals, respectively, at the beginning of time period n.

(a) If there are i infectious individuals at the beginning of a time period, what is the probability that a specified noninfected individual will become infected in that period?

(b) Is $\{X_n, n \geq 0\}$ a Markov chain? If so, give its transition probabilities.

(c) Is $\{Y_n, n \geq 0\}$ a Markov chain? If so, give its transition probabilities.

(d) Is $\{(X_n, Y_n), n \geq 0\}$ a Markov chain? If so, give its transition probabilities.

7. Consider a simple random walk on the integer points in which at each step a particle moves one step in the positive direction with probability p, one step in the negative direction with probability p, and remains in the same place with probability $q = 1 - 2p (0 < p < \frac{1}{2})$. Suppose an absorbing barrier is placed at the origin—that is, $P_{00} = 1$—and a reflecting barrier at N—that is, $P_{N,N-1} = 1$—and that the particle starts at n $(0 < n < N)$.

Show that the probability of absorption is 1, and find the mean number of steps.

8. A particle moves among n locations that are arranged in a circle (with the neighbors of location n being $n - 1$ and 1). At each step, it moves one position either in the clockwise position with probability p or in the counterclockwise position with probability $1 - p$.

(a) Find the transition probabilities of the reverse chain.

(b) Is the chain time reversible?

9. Suppose that a one-celled organism can be in one of two states—either A or B. An individual in state A will change to state B at an exponential rate α; an individual in state B divides into two new individuals of type A at an exponential rate β. Define an appropriate continuous-time Markov chain for a population of such organisms and determine the appropriate parameters for this model.

10. Consider a population in which each individual independently gives birth at an exponential rate λ and dies at an exponential rate μ. In addition, new members enter the population in accordance with a Poisson process with rate θ. Let $X(t)$ denote the population size at time t.

(a) What type of process is $\{X(t), t \geq 0\}$?

(b) What are its parameters?

(c) Find $E[X(t)|X(0) = i]$.

11. Consider a Markov chain $\{X_n, n \geq 0\}$ with $P_{NN} = 1$. Let $P(i)$ denote the probability that this chain eventually enters state N given that it starts in state i. Show that $\{P(X_n), n \geq 0\}$ is a martingale.

Sample Questions, Exam 2

1. (a) Prove the renewal equation

$$m(t) = F(t) + \int_0^t m(t-x)\, dF(x).$$

(b) Prove that the renewal function $m(t)$, $0 \le t < \infty$ uniquely determines the interarrival distribution F.

(c) Let $\{N(t), t \ge 0\}$ be a renewal process and suppose that for all n and t, conditional on the event that $N(t) = n$, the event times $S_1, \ldots, S_n$ are distributed as the order statistics of a set of independent uniform $(0, t)$ random variables. Show that $\{N(t), t \ge 0\}$ is a Poisson process. (*Hint:* Consider $E[N(s) \mid N(t)]$ and then use the result of

2. An equation of the form

$$g(t) = h(t) + \int_0^t g(t-x)\, dF(x)$$

is called a renewal-type equation. In convolution notation the above states that

$$g = h + g * F.$$

Either iterate the above or use Laplace transforms to show that a renewal-type equation has the solution

$$g(t) = h(t) + \int_0^t h(t-x)\, dm(x),$$

where $m(x) = \sum_{n=1}^{\infty} F_n(x)$. If h is directly Riemann integrable and F nonlattice with finite mean, one can then apply the key renewal theorem to obtain

$$\lim_{t\to\infty} g(t) = \frac{\int_0^\infty h(t)\, dt}{\int_0^\infty \bar{F}(t)\, dt}.$$

Renewal-type equations for $g(t)$ are obtained by conditioning on the time at which the process probabilistically starts over. Obtain a renewal-type equation for:

(a) $P(t)$, the probability an alternating renewal process is on at time t;

(b) $g(t) = E[A(t)]$, the expected age of a renewal process at t.

Apply the key renewal theorem to obtain the limiting values in (a) and (b).

3. Draw cards one at a time, with replacement, from a standard deck of playing cards. Find the expected number of draws until four successive cards of the same suit appear.

4. Let $\{X_n, n \geq 1\}$ denote an irreducible Markov chain having a countable state space. Now consider a new stochastic process $\{Y_n, n \geq 0\}$ that only accepts values of the Markov chain that are between 0 and N. That is, we define Y_n to be the nth value of the Markov chain that is between 0 and N. For instance, if $N = 3$ and $X_1 = 1$, $X_2 = 3$, $X_3 = 5$, $X_4 = 6$, $X_5 = 2$, then $Y_1 = 1$, $Y_2 = 3$, $Y_3 = 2$.

(a) Is $\{Y_n, n \geq 0\}$ a Markov chain? Explain briefly.

(b) Let π_j denote the proportion of time that $\{X_n, n \geq 1\}$ is in state j. If $\pi_j > 0$ for all j, what proportion of time is $\{Y_n, n \geq 0\}$ in each of the states $0, 1, \ldots, N$?

(c) Suppose $\{X_n\}$ is null recurrent and let $\pi_i(N)$, $i = 0, 1, \ldots, N$ denote the long-run proportions for $\{Y_n, n \geq 0\}$. Show that

$$\pi_j(N) = \pi_i(N) E[\text{time the } X \text{ process spends in } j \text{ between returns to } i], \quad j \neq i.$$

(d) Use (c) to argue that in a symmetric random walk the expected number of visits to state i before returning to the origin equals 1.

(e) If $\{X_n, n \geq 0\}$ is time reversible, show that $\{Y_n, n \geq 0\}$ is also.

5. Each individual in a biological population is assumed to give birth at an exponential rate λ and to die at an exponential rate μ. In addition, there is an exponential rate of increase θ due to immigration. However, immigration is not allowed when the population size is N or larger.

(a) Set this up as a birth and death model.

(b) If $N = 3$, $1 = \theta = \lambda$, $\mu = 2$, determine the proportion of time that immigration is restricted.

6. N customers move about among r servers. The service times at server i are exponential at rate μ_i and when a customer leaves server i it joins the queue (if there are any waiting—or else it enters service) at server j, $j \neq i$, with probability $1/(r-1)$. Let the state be $(n_1, \ldots, n_r)$ when there are n_i customers at server i, $i = 1, \ldots, r$. Show the corresponding continuous-time Markov chain is time reversible and find the limiting probabilities.

7. Let Z_n, $n \geq 1$, be a sequence of random variables such that $Z_1 = 1$ and given $Z_1, \ldots, Z_{n-1}$, Z_n is a Poisson random variable with mean Z_{n-1}, $n > 1$. What can we say about Z_n for n large?