Solution to Assignment 9

9.12: The likelihood function is $L(\theta) = \theta^n \exp(-n\theta\overline{X})$, maximized at $\hat{\theta} = 1/\overline{X}$. So

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \theta_0^n \overline{X}^n \exp(-n\theta_0\overline{X} + n) = \left[e\theta_0\overline{X}e^{-\theta_0\overline{X}}\right]^n,$$

and $\Lambda \le k$ if and only if

$$\overline{X}e^{-\theta_0\overline{X}} \le \frac{\sqrt[n]{k}}{e\theta_0} \stackrel{\text{def}}{=} c.$$

9.20: The best test, from the Neyman-Pearson Lemma, rejects H_0 if $X \ge 0.9$, and has power $P(X \ge 0.9|H_1) = \int_{0.9}^{1} 2x\,dx = 19\%$. This is the largest possible power.

9.22: The acceptance region is

$$\begin{aligned} A(\sigma_0^2) &= \{\mathbf{X} : \sigma_0^2 \in C(\mathbf{X})\} \\ &= \left\{\mathbf{X} : \sigma_0^2 \in \left(\frac{n\hat{\sigma}^2}{\chi^2_{n-1}(\alpha/2)}, \frac{n\hat{\sigma}^2}{\chi^2_{n-1}(1-\alpha/2)}\right)\right\} \\ &= \left\{\mathbf{X} : \hat{\sigma}^2 \in \left(\frac{\sigma_0^2\chi^2_{n-1}(1-\alpha/2)}{n}, \frac{\sigma_0^2\chi^2_{n-1}(\alpha/2)}{n}\right)\right\} \end{aligned}$$

The rejection region when $\sigma_0 = 1$, $n = 15$, and $\alpha = 5\%$ is

$$\left\{\mathbf{X} : \hat{\sigma}^2 \notin (0.375, 1.74)\right\}.$$