Stat 426—Practice Final Solution

(1) The marginal density of X is

$$f_X(x) = \int_0^\infty e^{-\theta} 2x\theta e^{-\theta x^2}\, d\theta = \frac{2x}{(1+x^2)^2},$$

and so

$$f_{\Theta|X}(\theta|x) = (1+x^2)^2 \theta e^{-(1+x^2)\theta}, \qquad \theta > 0.$$

The posterior mean is

$$\int_0^\infty (1+x^2)^2 \theta^2 e^{-(1+x^2)\theta}\, d\theta = \frac{2}{1+x^2}.$$

Setting

$$\frac{\partial}{\partial\theta} \log f_{\Theta|X}(\theta|x) = \frac{1}{\theta} - (1+x^2)$$

to zero, the mode of the posterior density is $1/(1+x^2)$, exactly half the posterior mean.

(2) The joint density is

$$\prod_{i=1}^n f(x_i|\theta) = \theta^n \exp\left(-\theta \sum_{i=1}^n (1/x_i)\right) \Big/ \prod_{i=1}^n x_i^2,$$

and $T = \sum_{i=1}^n (1/X_i)$ is sufficient by the factorization theorem.

(3) The likelihood ratio $f_0(X)/f_1(X)$ is $2/(3X)$, and the likelihood ratio test will reject H_0 if $2/(3X) < k$, or if $X > 2/(3k) \overset{\text{def}}{=} \tilde{k}$. The significance level of this test is

$$\alpha = P_{H_0}(X > \tilde{k}) = \int_{\tilde{k}}^1 2x\, dx = 1 - \tilde{k}^2.$$

Solving $1 - \tilde{k}^2 = 19\%$, $\tilde{k} = .9$ gives $\alpha = 19\%$, so the best test with this significance level rejects H_0 if $X > .9$. The power of this test is

$$P_{H_1}(X > .9) = \int_{.9}^1 3x^2\, dx = 27.1\%.$$

(4) The likelihood ratio statistic is

$$\Lambda = \frac{f(X|0)}{\max_{\theta\in\mathbb{R}} f(X|\theta)} = e^{-|X|},$$

so the generalized likelihood ratio test will reject H_0 if $|X| > k$. The level of this test is

$$\alpha = P_0(|X| > k) = \int_{|x|>k} \frac{1}{2} e^{-|x|}\, dx = e^{-k}.$$

To have $\alpha = 1/e$ we need $k = 1$, so the test rejects if $|X| > 1$. The power of the test if $\theta = 1$ is

$$P_1(|X| > 1) = \int_{|x|>1} \frac{1}{2} e^{-|x-1|}\, dx = \frac{e^2+1}{2e^2}.$$

(5) The maximum likelihood for p under the null hypothesis is $\overline{X}/4 = 40\%$. So $E_i = 25\binom{4}{i}(.4)^i(.6)^{4-i}$, or 3.24, 8.64, 8.64, 3.84, and 0.64 for $i = 0, \dots, 4$, respectively. So

$$-2\log\Lambda = 2\sum_{i=0}^{4} O_i \log(O_i/E_i) = 10.5.$$

The 95-th percentile for the chi-square distribution on 3 degrees of freedom is 7.81, so we would reject H_0.

(6) The pooled sample variance is $s_p^2 = (0.061 + 0.100)/2 = 0.0805$, so

$$s^2_{\overline{X}-\overline{Y}} = s_p^2(1/10 + 1/10) = 0.0161 = (0.127)^2.$$

The test statistic to test whether the means are equal is

$$T = \frac{\overline{X} - \overline{Y}}{s_{\overline{X}-\overline{Y}}} = 9.142.$$

Comparing $|T|$ with $t_{18}(0.025) = 2.101$, we would (easily) reject H_0 when $\alpha = 5\%$. The confidence interval is

$$(\overline{X} - \overline{Y} \pm t_{18}(0.05)s_{\overline{X}-\overline{Y}}) = (1.16 \pm 0.22).$$

(7) The 95th percentile for the F-distribution on 3 and 16 degrees is3.24. Since the F-statistic in the ANOVA table below exceeds this quantile, we would reject the null hypothesis that detergent does not effect suds.

Source	df	SS	MS	F
Suds	3	111.2	37.07	7.34
Error	16	80.8	5.05	
Total	19	192.0		

(8) Since $\overline{x} = 3$, $\overline{Y} = 6$, $\sum(x_i - \overline{x})^2 = 10$ and $\sum(x_i - \overline{x})(Y_i - \overline{Y}) = 20$,

$$\hat{\beta} = \frac{\sum(x_i - \overline{x})(Y_i - \overline{Y})}{\sum(x_i - \overline{x})^2} = 2$$

and

$$\hat{\alpha} = \overline{Y} - \hat{\beta}\overline{x} = 0.$$

Then $\sum e_i^2 = \sum(Y_i - \hat{\alpha} - \hat{\beta}x_i)^2 = 10$ and $s^2 = 10/3 = 3.33$. Using this,

$$s_{\hat{\beta}}^2 = \frac{s^2}{\sum(x_i - \overline{x})^2} = 0.333,$$

and $s_{\hat{\beta}} = 0.577$. The 95% confidence interval for β is

$$(\hat{\beta} \pm t_3(0.025)s_{\hat{\beta}}) = (1 \pm 1.837).$$