Linear Algebra

Vectors

- A column vector is a list of numbers stored vertically. The dimension of a column vector is the number of values in the vector. W is a 3-dimensional column vector and V is a 5-dimensional column vector:

$$W = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \qquad V = \begin{pmatrix} -1 \\ 0 \\ 4 \\ 2 \\ -5 \end{pmatrix}.$$

  Subscripts are used to denote single elements of the vector. For example, $W_2 = -1$, $V_4 = 2$.

- A row vector is a list of numbers stored horizontally. The dimension of a row vector is the number of values in the vector. U is a 4-dimensional row vector and T is a 2-dimensional row-vector:

$$U = \begin{pmatrix} 2 & -4 & 0 & 0 \end{pmatrix}.$$

$$T = \begin{pmatrix} 4 & 3 \end{pmatrix}.$$

  Elements are accessed by subscript: $U_3 = 0$, $T_1 = 4$.

- The term vector is used to refer to either a row vector or a column vector in situations where it doesn't matter whether the values are stored in a column or in a row.

- Two vectors with a common dimension can be added or subtracted:

$$\begin{pmatrix} 2 & -4 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 & -3 \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 & -3 \end{pmatrix}.$$

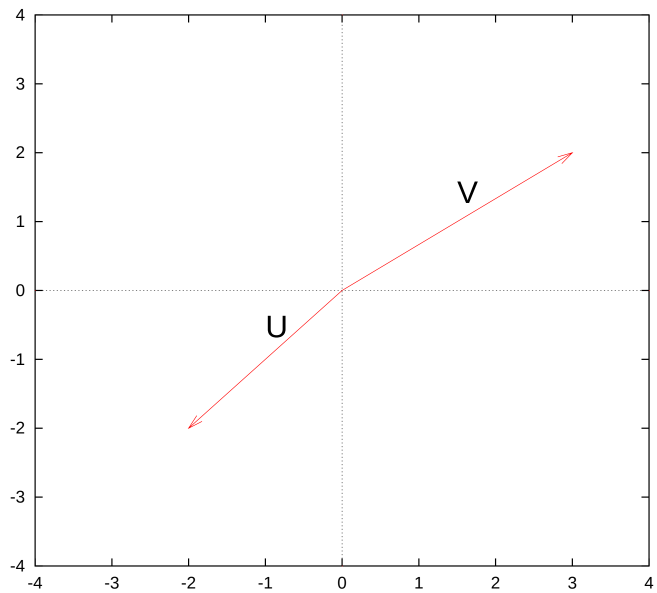
  Any vector can be multiplied by a scalar coefficient:

$$2 \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix}.$$

  Combining addition and scalar multiplication gives a linear combination:

$$3 \begin{pmatrix} 4 & 3 \end{pmatrix} - 2 \begin{pmatrix} 1 & -1 \end{pmatrix} = \begin{pmatrix} 10 & 11 \end{pmatrix}.$$

- Geometrically, a vector V can be viewed as a line segment in d-dimensional space with its tail at the origin and its head at the point V.



V is the vector (3,2), and U is the vector (-2,-2).

- The scalar product (also called the dot product or inner product) is formed from two vectors V and W having the same dimension. The scalar product is a single number (a "scalar"), and is notated $V \cdot W$ or $\langle V, W \rangle$. The definition of the scalar product is

  $$V \cdot W = \sum_i V_i W_i.$$

  For example, if $V = (3, 1, -1)$ and $W = (-1, 0, 1)$,

  $$V \cdot W = 3 \cdot -1 + 1 \cdot 0 + -1 \cdot 1 = -4.$$

  If $U = (4, -1, 0, 4)$ and $T = (0, -1, -1, 2)$,

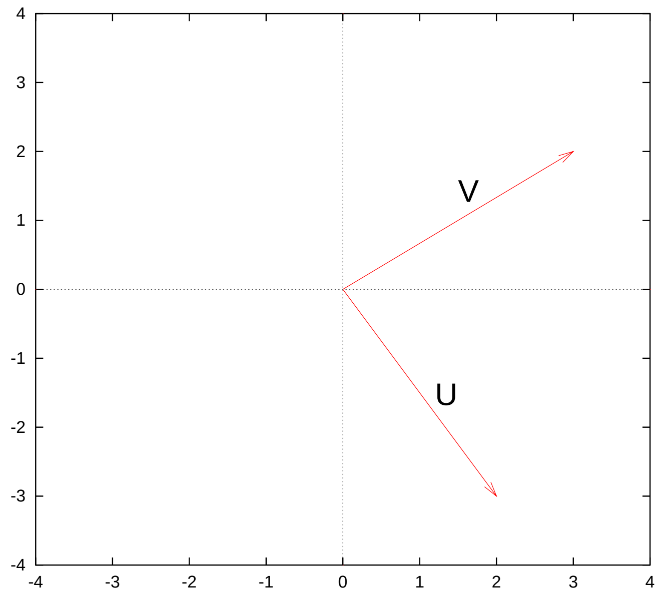
  $$U \cdot T = 4 \cdot 0 + -1 \cdot -1 + 0 \cdot -1 + 4 \cdot 2 = 9.$$

- If two d-dimensional vectors U and V have inner product 0, $\langle U, V \rangle = 0$, then U and V are orthogonal, or perpendicular.

  Viewing 2-dimensional vectors as points in the plane,

  $$V = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad U = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix},$$

  then $\langle V, U \rangle = 0$ when V and U are perpendicular in the geometric sense (the angle between them is $\pi/2$ radians).



V is the vector (3,2), and U is the vector (2,-3).

- A linear combination of d-dimensional vectors $V_1, V_2, \dots, V_m$ is an expression of the form

$$c_1 V_1 + c_2 V_2 + \cdots + c_m V_m,$$

where $c_1, \dots, c_m$ are scalars called coefficients. For example, if

$$V_1 = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix},$$

and $c_1 = 2$, $c_2 = -1$, $c_3 = 3$, then $c_1 V_1 + c_2 V_2 + c_3 V_3$ is

$$\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

- A set of d-dimensional vectors $V_1, \dots, V_m$ are linearly dependent if there exist scalars $c_1, \dots, c_m$, not all of which are zero, such that the linear combination $c_1 V_1 + \cdots + c_m V_m$ is zero. For example,

$$V_1 = \begin{pmatrix} 3 \\ 1 \\ -3 \\ 2 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix}, \quad V_3 = \begin{pmatrix} -6 \\ 1 \\ 3 \\ 2 \end{pmatrix},$$

are linearly dependent, since if $c_1 = -2$, $c_2 = 3$, and $c_3 = -1$, then $c_1 V_1 + c_2 V_2 + c_3 V_3 = 0$.
- A set of d-dimensional vectors $V_1, \dots, V_m$ are linearly independent if they are not linearly dependent. That is, whenever $c_1 V_1 + \cdots + c_m V_m = 0$, then $c_1 = \cdots = c_m = 0$.
For example, suppose

$$V_1 = \begin{pmatrix} 3 \\ 0 \\ -3 \\ 2 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 5 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \quad V_3 = \begin{pmatrix} -6 \\ -2 \\ 3 \\ 2 \end{pmatrix},$$

and $c_1 V_1 + c_2 V_2 + c_3 V_3 = 0$. The linear combination can be written

$$\begin{pmatrix} 3c_1 + 5c_2 - 6c_3 \\ -2c_3 \\ -3c_1 - c_2 + 3c_3 \\ 2c_1 + 2c_2 + 2c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

From the second line, $c_3 = 0$, and from the final line $c_2 = -c_1$. From the third line, $c_2 = -3c_1$, so we conclude $c_1 = c_2 = 0$, hence V_1, V_2, V_3 are linearly independent.

- It is a fact that any set of $m > d$ d-dimensional vectors must be linearly dependent. For example, there can never be a set of 4 linearly independent vectors having dimension 3.

Matrices

- A $m \times n$ matrix A is a $m \times n$ array of numbers, where M_{ij} refers to the value in the i^{th} row and j^{th} column. For example, the following is a 2×3 matrix

$$A = \begin{pmatrix} 4 & 3 & 0 \\ -1 & 0 & -2 \end{pmatrix},$$

where $A_{12} = 3$, $A_{22} = 0$, etc.
The following is a 3×2 matrix

$$B = \begin{pmatrix} 1 & -2 \\ -1 & 3 \\ 2 & 0 \end{pmatrix},$$

where $B_{11} = 1$, $B_{32} = 0$, etc.

- Matrices of the same shape can be added and subtracted:

$$\begin{pmatrix} 4 & 3 & 0 \\ -1 & 0 & -2 \end{pmatrix} + \begin{pmatrix} 0 & -2 & 3 \\ 4 & 4 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 3 \\ 3 & 4 & -3 \end{pmatrix},$$

Any matrix can be multiplied by a scalar:

$$3 \begin{pmatrix} 4 & 3 & 0 \\ -1 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 12 & 9 & 0 \\ -3 & 0 & -6 \end{pmatrix},$$

We can form linear combinations of matrices:

$$2 \begin{pmatrix} 2 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix} - 3 \begin{pmatrix} 1 & 2 & 0 \\ -1 & 2 & -3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -2 \\ 3 & -6 & 17 \end{pmatrix}.$$

- A n-dimensional column vector is also a $n \times 1$ matrix. A n-dimensional row vector is also a $1 \times n$ matrix.
- The transpose of an $m \times n$ matrix A, written A', is an $n \times m$ matrix where $A'_{ij} = A_{ji}$. For example,

$$\begin{pmatrix} 4 & 3 & 0 \\ -1 & 0 & -2 \end{pmatrix}' = \begin{pmatrix} 4 & -1 \\ 3 & 0 \\ 0 & -2 \end{pmatrix},$$

  A matrix A is symmetric if $A = A'$. For example,

$$S = \begin{pmatrix} 4 & 3 \\ 3 & 5 \end{pmatrix}$$

  is symmetric, while

$$T = \begin{pmatrix} 4 & 3 \\ 6 & 5 \end{pmatrix}$$

  is not.
- If A is a $m \times n$ matrix and V is a n-dimensional column vector, we can form the matrix vector product $W = AV$, where W_i is the dot product of V with row i of A. For example,

$$\begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \cdot 2 + -1 \cdot -1 \\ 3 \cdot 1 + -1 \cdot 4 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \end{pmatrix}.$$

- The nullspace of M, written $\text{Null}(M)$, is the set of all vectors V such that $MV = 0$. For example, if

$$\begin{pmatrix} 2 & -1 & 4 \\ 0 & 1 & 2 \end{pmatrix}$$

then

$$V = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

is in $\text{Null}(M)$.
The zero vector is always in $\text{Null}(M)$, and if the columns of M are linearly independent, the zero vector is the only vector in $\text{Null}(M)$. But if the columns of M are linearly dependent, there will be infinitely many nonzero vectors in $\text{Null}(M)$.

A square $(m \times m)$ matrix with nullspace containing only the zero vector is nonsingular, otherwise it is singular. Equivalently, a square matrix is nonsingular if and only if its columns are linearly independent.

- If A is a $m \times n$ matrix and B is a $n \times r$ matrix, we can form the matrix matrix product $C = AB$, where C is a $m \times r$ matrix whose elements are defined as: C_{ij} is the dot product of row i of A with column j of B. For example,

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 7 & -3 \\ 11 & -5 \end{pmatrix},$$

where, for example, $11 = 1 \cdot 3 + 0 \cdot 1 + 4 \cdot 2$.

Rectangular matrices can only be multiplied if the number of columns in the first matrix is equal to the number of rows in the second matrix (i.e. AB can be formed only if A is $m \times n$ and B is $n \times r$).

For square matrices, the products AB and BA can both be formed. However it is important to note that they are different:

$$\begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ 7 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 5 & -7 \\ 2 & -1 \end{pmatrix}.$$

- For any matrix $m \times n$ matrix A, the products $A'A$ and AA' can always be formed. The first product is $n \times n$ and the second product is $m \times m$.

  The product $A'A$ is the "column-wise inner product matrix", since $(A'A)_{ij}$ is the inner product of the i^{th} column of A with the j^{th} column of A.

  The product AA' is the "row-wise inner product matrix", since $(AA')_{ij}$ is the inner product of the i^{th} row of A with the j^{th} row of A.

For example, if

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 4 \end{pmatrix},$$

then

$$A'A = \begin{pmatrix} 5 & -2 & 6 \\ -2 & 1 & -1 \\ 6 & -1 & 17 \end{pmatrix},$$

and

$$AA' = \begin{pmatrix} 6 & 6 \\ 6 & 17 \end{pmatrix}.$$

Note that both $A'A$ and AA' are symmetric.

- The identity matrix is a special square ($n \times n$) matrix I, where $I_{jj} = 1$ and $I_{ij} = 0$ if $i \neq j$. For example, the 4×4 identity matrix is

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The identity matrix acts like "1" for matrix multiplication: $AI = A$ and $IA = A$ (if A is $m \times n$, the first I is the $m \times m$ identity matrix, and the second I is the $n \times n$ identity matrix).

- If A is a square $(n \times n)$ matrix with linearly independent columns, then an $n \times n$ matrix A^{-1} can be constructed such that $AA^{-1} = A^{-1}A = I$, where I is the $n \times n$ identity.

  For example,

$$A = \begin{pmatrix} 3 & 2 \\ 7 & 1 \end{pmatrix} \quad A^{-1} = \begin{pmatrix} -1/11 & 2/11 \\ 7/11 & -3/11 \end{pmatrix}$$

  For a 2×2 matrix, the general form of the inverse is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

  where if $ad = bc$, A is singular and has no inverse.

  For $d > 2$, the formula for the inverse of a $d \times d$ matrix is very complicated, but inverses can be easily calculated on a computer.