Statistics 403 Problem Set 2

Due in lab on Friday, September 25th

1. Suppose we have an iid sample $X_1, \ldots, X_n$ and wish to estimate EX as accurately as possible. One alternative is to double the sample size to $2n$. Another alternative available to us is to reduce the variance $\sigma^2 = \text{var}(X)$ to a new value $f\sigma^2$, where $f < 1$ is a constant. Describe the values of f that will give a more precise estimate of EX when taking the second alternative over the first.

   Solution: If we follow the first alternative, the variance of $\bar{X}$ will be $\sigma^2/2n$. If we follow the second alternative, the variance will be $f\sigma^2/n$. For the second alternative to give the more precise estimate of EX, we need $f\sigma^2/n < \sigma^2/2n$, so we must have $f < 1/2$.

2. Suppose we have a sample $X_1, \ldots, X_{10}$ from a normal population with mean one and standard deviation three. What is the probability that $|\bar{X}|$ is greater than 2? To get a numerical answer, you will need to use a normal probability table, or use R.

   Solution:

$$\begin{aligned} P(|\bar{X}| > 2) &= P(\bar{X} > 2) + P(\bar{X} < -2) \\ &= P(\frac{\bar{X}-1}{3/\sqrt{10}} > \frac{2-1}{3/\sqrt{10}}) + P(\frac{\bar{X}-1}{3/\sqrt{10}} < \frac{-2-1}{3/\sqrt{10}}) \\ &= P(Z > \sqrt{10}/3) + P(Z < -\sqrt{10}) \\ &\approx 0.15. \end{aligned}$$

3. Suppose we have a dataset containing three values, 1, 2, c, where $c > 0$ is a constant. Derive simplified expressions for the arithmetic mean (i.e. the usual average value) and the geometric mean. Show that the geometric mean is always less than the arithmetic mean (hint: write an expression for the arithmetic mean minus the geometric mean, and use calculus to show that its minimum value is positive).

   Solution:

   The arithmetic mean is

$$\frac{1+2+c}{3} = 1 + c/3,$$

   and the geometric mean is

$$(1 \times 2 \times c)^{1/3} = 2^{1/3} \times c^{1/3}.$$

The arithmetic mean minus the geometric mean is

$$1 + c/3 - 2^{1/3}c^{1/3}.$$

Differentiating this with respect to c and setting the value to zero, we get

$$1/3 - 2^{1/3}c^{-2/3}/3 = 0,$$

so $c = \sqrt{2} \approx 1.41$. The second derivative is $2^{4/3}c^{-5/3}/9$ which is always positive if $c > 0$. Thus $c = \sqrt{2}$ is a local minimum. We can check directly that when $c = \sqrt{2}$ the arithmetic mean is around 1.47 and the geometric mean is around 1.41. We should also check that at $c = 0$ the arithmetic mean is greater than the geometric mean (which is zero in that case). Since the arithmetic mean is greater than the geometric mean at the interior minimum value, and at the boundary value, it must hold for all values of c.

4. Suppose we observe 10 data points with an average value of 6 and a standard deviation of 2. We then add three additional data points to our data set, each of which is a "6." (i) What is the average value of the new data set of 13 data points? (ii) What is the standard deviation of the new data set of 13 data points?

   Solution: For part (i), the average value is

$$\begin{aligned}
\frac{X_1 + \cdots + X_{10} + 6 + 6 + 6}{13} &= \frac{X_1 + \cdots + X_{10}}{13} + 18/13 \\
&= \frac{X_1 + \cdots + X_{10}}{10} \cdot \frac{10}{13} + 18/13 \\
&= 6 \cdot \frac{10}{13} + 18/13 \\
&= 78/13 \\
&= 6.
\end{aligned}$$

5. Suppose we flip two coins. The coins are fair so each has probability 1/2 of landing as a head and probability 1/2 of landing as a tail. Let $A = 1$ if the first coin lands as a head and $A = 0$ if it lands as a tail. Similarly, let $B = 1$ if the second coin lands as a head and $B = 0$ otherwise.

   (a) What is the variance of A?

   Solution: The expected value of A is

$$EA = 1 \cdot 1/2 + 0 \cdot 1/2 = 1/2.$$

The variance of A is

$$(1 - 1/2)^2/2 + (0 - 1/2)^2/2 = 1/4.$$

(b) If we play a game in which the coins are flipped independently and the "score" is $A + B$, what are the expected value and variance of the score?

Solution: We know that expected values are always additive, and variance are additive when the random variables are independent. Thus in this case we can add both the expected values and the variances:

$$E(A + B) = EA + EB = 1/2 + 1/2 = 1.$$

$$\text{var}(A + B) = \text{var}(A) + \text{var}(B) = 1/4 + 1/4 = 1/2.$$

(c) Now suppose the coins are linked in some way so that A always equals B, and their common value has probability 1/2 of being a head, and probability 1/2 of being a tail. What are the mean and variance of $A + B$?

Solution: Since the expected value is additive regardless of whether the random variables are independent, the expected value of $A+B$ is still 1 here. The sample space is $\{0, 2\}$, and we have probability 1/2 of observing either of these values. Thus the variance of $A + B$ is

$$(2 - 1)^2/2 + (0 - 1)^2/2 = 1.$$

(d) Now suppose the coins are linked in some way so that A never equals B, but A still has probability 1/2 of being a head, and probability 1/2 of being a tail. What are the mean and variance of $A + B$?

Solution: The expected value of $A + B$ is still 1 here. The sample space is just the value $\{1\}$, hence the variance is zero.