Statistics 403 Problem Set 6

Due in lab on Friday, October 23rd

1. Practice calculating probabilities and quantiles:

   (a) Suppose X is normal with mean 3 and variance 1. Find the 75^{th} percentile of X.

   Solution: The quantile is the value q that satisfies the equation

   $$P(X \leq q) = 0.75$$

   To solve for q, follow these steps

   $$\begin{aligned} P(X \leq q) &= P(X - 3 \leq q - 3) \\ &= P(Z \leq q - 3) \\ &= 0.75. \end{aligned}$$

   The 75^{th} percentile of a standard normal distribution is 0.67, so we solve $q - 3 = 0.67$ to get $q = 3.67$.

   (b) Suppose X is normal with mean 2 and variance 4. Find the probability that X is less than 5.

   Solution:

   $$\begin{aligned} P(X \leq 5) &= P((X - 2)/2 \leq (5 - 2)/2) \\ &= P(Z \leq 1.5) \\ &= 0.93. \end{aligned}$$

   (c) Suppose X is normal with mean 1 and standard deviation 2. Find the point f such that X has probability 0.35 of being greater than f.

   Solution: We need to solve

   $$\begin{aligned} 0.35 &= P(X \geq f) \\ &= P((X - 1)/2 \geq (f - 1)/2) \\ &= P(Z \geq (f - 1)/2) \end{aligned}$$

   Therefore, $P(Z \leq (f - 1)/2) = 1 - 0.35 = 0.65$. The 65^{th} percentile of the standard normal distribution is 0.39. Thus we solve $(f - 1)/2 = 0.39$ to get $f = 1.78$.

(d) Suppose X is normal with mean 1 and standard deviation 1. Find the probability that X is between 0 and 1.5.

Solution:

$$\begin{aligned} P(0 \leq X \leq 1.5) &= P(-1 \leq X - 1 \leq 0.5) \\ &= P(-1 \leq Z \leq 0.5) \\ &= P(Z \leq 0.5) - P(Z \leq -1) \\ &= 0.691 - 0.159 \\ &= 0.53. \end{aligned}$$

(e) Suppose X is normal with mean 3 and standard deviation 3. Find the 20^{th} percentile of X.

Solution:

$$\begin{aligned} 0.2 &= P(X \leq Q) \\ &= P((X-3)/3 \leq (Q-3)/3) \\ &= P(Z \leq (Q-3)/3) \end{aligned}$$

The 20^{th} percentile of a standard normal random variable is -0.84, so we solve $(Q-3)/3 = -0.84$ to get $Q = 0.48$.

(f) Suppose X is normal with mean 5 and standard deviation 2. Find the "interdecile range" of X (the distance from the 10^{th} percentile of X to the 90^{th} percentile of X).

Solution: To get the 90^{th} percentile:

$$\begin{aligned} 0.9 &= P(X \leq Q_1) \\ &= P((X-5)/2 \leq (Q_1-5)/2) \\ &= P(Z \leq (Q_1-5)/2) \end{aligned}$$

The 90^{th} percentile of a standard normal value is 1.28, so we solve $(Q_1 - 5)/2 = 1.28$ to get $Q_1 = 7.56$.

To get the 10^{th} percentile:

$$
\begin{aligned}
0.1 &= P(X \le Q_2) \\
&= P((X-5)/2 \le (Q_2-5)/2) \\
&= P(Z \le (Q_2-5)/2)
\end{aligned}
$$

The 10^{th} percentile of a standard normal value is -1.28, so we solve $(Q_2-5)/2 = -1.28$ to get $Q_2 = 2.44$.

Thus the interdecile range is 5.12.

2. Suppose we make the assumption that a particular instrument yields measurements that are unbiased for our quantity of interest, μ, and have standard deviation σ. The measurements are unbiased, but the instrument is actually less precise than thought, as it has a standard deviation of $f\sigma$ for a constant $f > 1$. If we form a 95% confidence interval for μ based on our incorrect assumptions about the operating characteristics of the instrument, what will its actual coverage probability be?

   Solution: The interval we will use is $\bar{X} \pm 2\sigma/\sqrt{n}$. The coverage probability of this interval can be rewritten as

$$
\begin{aligned}
P(\bar{X} - 2\sigma/\sqrt{n} \le \mu \le \bar{X} + 2\sigma/\sqrt{n}) &= P(-2\sigma/\sqrt{n} \le \mu - \bar{X} \le 2\sigma/\sqrt{n}) \\
&= P(-2\sigma/\sqrt{n} \le \bar{X} - \mu \le 2\sigma/\sqrt{n}).
\end{aligned}
$$

   The standard deviation of $\bar{X}$ is $f\sigma/\sqrt{n}$. So to complete the standardization, we have

$$
P(-2/f \le (\bar{X} - \mu)/(f\sigma/\sqrt{n}) \le 2/f) = P(-2/f \le Z \le 2/f).
$$

   Thus the coverage probability is $P(-2/f \le Z \le 2/f)$. Note that if $f = 1$ we get 95% coverage as desired. If $f > 1$ we get lower coverage than desired, and if $f < 1$ we get higher coverage than desired.