Probability Models

Important Concepts

Read Chapter 2

- Probability Models
- Examples
  - The Classical Model
  - Discrete Spaces
- Elementary Consequences of the Axioms
- The Inclusion Exclusion Formulas
- Some Indiscrete Models
- Monotone Sequences and Continuity

Experiments

Phenomena

- Unpredictable in detail
- The set of possible outcomes in known.

Examples a) Scientific experiments

b) Games of chance

c) Human performace

d) Financial indices

e) The Weather

Events and The Sample Space

The Sample Space. Let Ω denote the set of possible outcomes for a given experiment.

Events: Susbsets of the sample space, $A, B, C \subseteq \Omega$.

Example: *Coin Tossing.* $\Omega = \{hH, hT, tH, tT\}$ and $A = \{hT, tH\}$.

The Algebra of Events Set theory operations on events–for example,

$$A \cup B = \{\omega : \omega \in A \text{ or } \omega \in B\},$$

$$AB = \{\omega : \omega \in A \text{ and } \omega \in B\},$$

$$A^c = \{\omega : \omega \notin A\}.$$

$$B - A = BA^c$$

The Model

Three Elements

- The sample space: $\Omega \neq \emptyset$.
- Events: Subsets of $A, B, C, \dots \subseteq \Omega$.
- Probability: Let $\mathcal{A}$ be the class of events,

and let $P : \mathcal{A} \to I\!R$ must satisfy

$$P(\Omega) = 1, \tag{1}$$

$$0 \leq P(A) \leq 1, \tag{2}$$

$$P(A \cup B) = P(A) + P(B) \tag{3}$$

whenever A and B are events for which $AB = \emptyset$.

Notes a). Probability is a property of events.

b). (1), (2), and (3) are axioms and admit various interpretations.

The Classical Model

Games of Chance

The Model. Ω is a finite set; $\mathcal{A}$ is the class of all subsets of Ω; and

$$P(A) = \frac{\#A}{\#\Omega}.$$

Example: Roulette

$$\Omega = \{\mathbf{0}, \mathbf{00}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \cdots, \mathbf{35}, \mathbf{36}\}$$

and

$$P(\{\text{Red Outcome}\}) = \frac{18}{38} = \frac{9}{19}.$$

The Birthday Problem

Q: If n people gather, what is the probability that no two have the same birthday?

A: Regard the birthdays of the n people as a sample *w.r.* from $\{1, 2, \cdots, 365\}$ (ignoring leap year). Then Ω is all lists

$$\omega = (i_1, \cdots, i_n)$$

and $\#\Omega = 365^n$. Let

$$A = \{\omega : i_j \neq i_k \text{ all } j \neq k\}.$$

Then A consists of all permutations of n days, $\#A = (365)_n$, and

$$P(A) = \frac{(365)_n}{365^n} = p_n \text{ say.}$$

Some Values

n	8	16	24	32	40
p_n	.924	.716	.462	.247	.109

Discrete Probability Models

Suppose $\Omega = \{\omega_1, \omega_2 \cdots\}$, finite or infinite; let

$$p : \Omega \to \mathbb{R},$$

satisfy

$$p(\omega) \geq 0 \text{ for all } \omega,$$
$$\sum_{\omega \in \Omega} p(\omega) = 1.$$

Let

$$P(E) = \sum_{\omega \in E} p(\omega)$$

for $E \subseteq \Omega$.

Notes a) Then (1), (2), and (3) hold.

b) $p(\omega) = P(\{\omega\})$.

Example. In the classical model, $p(\omega) = 1/\#\Omega$.

On Infinite Sums

If $x_1, x_2, \cdots \in \mathbb{R}$, then

$$\sum_{k=1}^{\infty} x_k = \lim_{n \to \infty} \sum_{k=1}^{n} x_k,$$

provided that the limit exists.

Examples a). If $-1 < x < 1$, then

$$\sum_{k=1}^{\infty} x^{k-1} = \frac{1}{1-x}.$$

b). For any x,

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k = e^x.$$

Alternative Notation: If $A = \{x_1, x_2, \cdots\}$, and $f : A \to [0, , \infty)$, write

$$\sum_{x \in A} f(x) = \sum_{k=1}^{\infty} f(x_k).$$

Waiting for Success

Play Roulette Until a You Win
Betting on Red

Let

$$r = \frac{9}{19},$$

$$q = 1 - r = \frac{10}{19},$$

and

$$\Omega = \{1, 2, \cdots\}$$

Then, intuitively,

$$p(1) = r,$$
$$p(2) = qr,$$
$$p(3) = q^2 r,$$
$$\cdots,$$
$$p(\omega) = rq^{\omega-1}.$$

Then

$$\sum_{\omega \in \Omega} p(\omega) = \sum_{\omega=1}^{\infty} rq^{\omega-1} = \frac{r}{1-q} = 1.$$

Let

$$P(A) = \sum_{\omega \in A} p(\omega).$$

Amusing Calculation: Let Odd $= \{1, 3, \cdots\}$. Then

$$P(\text{Odd}) = \sum_{k=0}^{\infty} rq^{(2k+1)-1} = r \sum_{k=0}^{\infty} q^{2k} = \frac{r}{1-q^2} = \frac{19}{29}.$$

What Does Probability Mean?

The Subjective Interpretation. Probabilities reflect the opinion of the observer.

Strategy: Assess probabilities by imagining bets.

Examples a). Peter is willing to give two to one odds that it will rain tomorrow. His subjective probability for rain tomorrow is at least 2/3.

b). Paul accepts the bet. His subjective probability for rain tomorrow is at most 1/3.

Applications Business.

The Objective Interpretation

Thought Experiment: Imagine the experiment repeated N times. For an event A, let

$$N_A = \# \text{ occurrences of } A.$$

Then

$$P(A) = \lim_{N \to \infty} \frac{N_A}{N}.$$

Example: Coin Tossing

N	N_H/N
100	.550
1000	.493
10000	.514
100000	.503

Note: Consistent with $P(H) = .5$.

Example. In many roulette games, about 9/19 will result in red.

Consequences of the Axioms

Suppose that P satisfies (1), (2), and (3).

If A and B are events for which $A \subseteq B$, then

$$P(B - A) = P(B) - P(A). \tag{4}$$

For any event A,

$$P(A^c) = 1 - P(A). \tag{5}$$

In particular,

$$P(\emptyset) = 0. \tag{6}$$

For any events A and B,

$$P(A \cup B) = P(A) + P(B) - P(AB) \tag{7}$$

If $A_1, \cdots, A_m$ are any m events, then

$$P(\bigcup_{i=1}^{m} A_i) \leq \sum_{i=1}^{m} P(A_i), \tag{8}$$

with equality if $A_i A_j = \emptyset$ whenever $i \neq j$.

Proofs

If $A \subseteq B$, then

$$B = A \cup (B - A)$$

and $A \cap (B - A) = \emptyset$. So,

$$P(B) = P(A) + P(B - A),$$

by (3) and, therefore,

$$P(B - A) = P(B) - P(A). \tag{4}$$

For (5), $A^c = \Omega - A$. So,

$$P(A^c) = P(\Omega) - P(A) = 1 - P(A). \tag{5}$$

For (6). $P(\emptyset) = P(\Omega^c) = 0$.

Example. In the birthday problem, the probability that at least two people have the same birthday is A^c, and

$$P(A^c) = 1 - P(A) = 1 - \frac{(365)_n}{365^n}.$$

Proofs: Continued

For (7),

$$A \cup B = A \cup (B - AB),$$

and $A \cap (B - AB) = \emptyset$. So,

$$\begin{aligned} P(A \cup B) &= P(A) + P(B - AB) \\ &= P(A) + P(B) - P(AB). \end{aligned}$$

If $m = 2$, then

$$P(\bigcup_{i=1}^{m} A_i) \leq \sum_{i=1}^{m} P(A_i), \tag{8}$$

by (7); and if $A_1 A_2 = \emptyset$, then there is equality by (3). The general case follows from mathematical induction.

More on Unions

If $A_1, \cdots, A_m$ are events, let

$$\begin{aligned} \sigma_1 &= \sum_{i=1}^{m} P(A_i), \\ \sigma_2 &= \sum_{1 \leq i < j \leq m} P(A_i A_j), \\ \sigma_3 &= \sum_{1 \leq i < j < k \leq m} P(A_i A_j A_k), \\ &\cdots, \\ \sigma_k &= \sum_{1 \leq i_1 \cdots < i_k \leq m} P(A_{i_1} \cdots A_{i_k}), \\ &\cdots, \\ \sigma_m &= P(A_1 A_2 \cdots A_m). \end{aligned}$$

Then

$$P(\bigcup_{i=1}^{m} A_i) = \sigma_1 - \sigma_2 + \cdots \pm \sigma_m.$$

Proof. By induction–messy.

The Matching Problem

Let Ω be all permuations

$$\omega = (i_1, \cdots, i_n)$$

of $1, 2, \cdots, n$. Thus,

$$\Omega = n!.$$

Let

$$A_j = \{\omega : i_j = j\}$$

$$A = \bigcup_{i=1}^{n} A_i.$$

Then

$$\sigma_k = \binom{n}{k} P(A_1 \cdots A_k),$$

by symmetry.

Examples. Gift exhange

Here

$$P(A_1) = \frac{1 \times (n-1)!}{n!} = \frac{1}{n!},$$

$$P(A_1 A_2) = \frac{(n-2)!}{n!} = \frac{1}{(n)_2},$$

$$\cdots,$$

$$P(A_1 \cdots A_k) = \frac{(n-k)!}{n!} = \frac{1}{(n)_k},$$

for $k = 1, \cdots, n$. So,

$$\sigma_k = \binom{n}{k}(n)_k = \frac{1}{k!},$$

$$P(A) = \sigma_1 - \sigma_2 + \cdots \pm \sigma_n$$

$$= \sum_{k=1}^{n} \frac{1}{k!}(-1)^{k-1},$$

and

$$P(A) = 1 - \sum_{k=0}^{n} \frac{1}{k!}(-1)^k \approx 1 - \frac{1}{e}.$$

Note: Accurate to three places if $n \geq 6$.

Refinements

More on Events. Not all subsets of Ω need be events; but the class of events must be closed under union, intersection, and complementation.

More on the Third Axiom. A stronger version of (3) requires

$$P(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} P(A_k), \qquad (3*)$$

whenever $A_1, A_2, \cdots$ are mutually exclusive events (that is, $A_i A_j = \emptyset$ for $i \neq j$).

Remark: (3*) implies (3).

Proposition. *The discrete probability models satisfy (3*), as well as (3).*

Proof. Omitted

Some Indiscrete Models

Intervals

$$(a, b) = \{x : a < x < b\},$$

$$(a, b] = \{x : a < x \leq b\},$$

$$[a, b) = \{x : a \leq x < b\},$$

$$[a, b] = \{x : a \leq x \leq b\},$$

Densities. Let Ω be an interval and f a function for which $f(\omega) \geq 0$ and

$$\int_{\Omega} f(\omega) d\omega = 1.$$

Then let

$$P(I) = \int_I f(\omega) d\omega$$

for intervals I and extend f to a larger class of events using the axioms.

Example *The Uniform Spinner*. Let $\Omega = (-\pi, \pi]$ and $f(\omega) = 1/2\pi$. Then

$$P((a, b)) = \cdots = P([a, b]) = \frac{b-a}{2\pi}.$$

Amusing Calculation

About the Extension Process

Note. For any ω,

$$P(\{\omega\}) = P([\omega, \omega]) = \int_{\omega}^{\omega} f(\omega')d\omega' = 0.$$

If

$$C = \{\omega_1, \omega_2, \cdots\},$$

then

$$P(C) = \sum_{i=1}^{\infty} P(\{\omega_i\}) = 0.$$

The probability of a rational outcome is zero.

Monotone Sequences

Events $A_1, A_2, \cdots$ are *increasing* if

$$A_1 \subseteq A_2 \subseteq \cdots$$

and *decreasing* if

$$A_1 \supseteq A_2 \supseteq \cdots .$$

The *limit* of an increasing (respectively, decreasing) sequence is

$$A_\infty = \bigcup_{k=1}^{\infty} A_k,$$

respectively,

$$A_\infty = \bigcap_{k=1}^{\infty} A_k.$$

Example. If $\Omega = I\!R$ and

$$A_k = (-\infty, \frac{1}{k}) = \{\omega : \omega < \frac{1}{k}\},$$

then A_k are decreasing and

$$A_\infty = \{\omega : \omega < \frac{1}{k} \text{ for all} k\}$$
$$= (-\infty, 0].$$

De Morgan's Laws. For any events $A_i,\ i = 1, \cdots, n$,

$$\Big(\bigcup_{i=1}^{n} A_i\Big)^c = \bigcap_{i=1}^{n} A_i^c,$$

$$\Big(\bigcap_{i=1}^{n} A_i\Big)^c = \bigcup_{i=1}^{n} A_i^c.$$

Also true if $n = \infty$. *Proof-e.g.*. $\omega \in (\cup_{i=1}^{n} A_i)^c$ iff $\omega \notin \cup_{i=1}^{n} A_i$ iff $\omega \notin A_i$ for any i iff $\omega \in cup_{i=1}^{n} A_i^c$.

Corollary. If $A_1, A_2, \cdots$ is increasing or decreasing, then then

$$(A_\infty)^c = (A^c)_\infty.$$

The Monotone Sequences Theorem

Suppose that P satisfies (1), (2), and (3°). Then P satisfies (3) iff

$$P(A_\infty) = \lim_{n \to \infty} P(A_n),$$

whenever $A_1, A_2, \cdots$ is an increasing, or decreasing, sequence of events.

Proof. Later, or see the text.

Remarks a). Type of continuity.

b). Equivalent to (3).

c). Useful.