

## Prologue

### Sampling Experiments

#### Box Ticket Model

- A (e.g., shoe) box.
- Contains (e.g. movie) tickets (the population).
- Shake the box.
- Remove some tickets (the sample).

#### Types of Samples

- Without replacement
- With replacement (and reshake the box).

## Ordered Samples

Label the tickets  $1, 2, \dots, N$ . If  $n$  tickets are drawn sequentially, then  $\Omega$  consists of list

$$\omega = (i_1, i_2, \dots, i_n),$$

where  $1 \leq i_1, \dots, i_n \leq N$  are the tickets drawn.

**Sampling With Replacement:**  $\Omega$  consists of all lists, and

$$\Omega = \#N^n.$$

**Sampling Without Replacement:**  $\Omega$  consists of all permutations, and

$$\Omega = (N)_n = N \times (N - 1) \times \dots \times (N - n + 1).$$

*Note* Requires  $n \leq N$ .

## An Example

*Q:* If a balanced die is rolled 6 times, what is the probability that every face appears?

*A:* Here  $n = N = 6$  and sampling with *w.r.*. So,

$$\#\Omega = 6^6.$$

The event  $E$  that every face appears

$$E = \{\text{all permutations of } 6\}.$$

So,

$$\#E = 6!,$$

and

$$P(E) = \frac{6!}{6^6} = .0154 \dots$$

## Conditional Probability And Independence

Important Concepts From  
Chapter 3

## Conditional Probability

If  $A$  and  $B$  are events and  $P(A) > 0$ , then

$$P(B|A) = \frac{P(AB)}{P(A)}$$

is called the *conditional probability of B given A*.

*Notes.*  $P(B|A)$  is a probability in  $B$ , but not  $A$ ;

For example,  $P(B^c|A) = 1 - P(B|A)$ .

*Note.* In the classical model,

$$P(A) = \#A/\#\Omega,$$

$$P(AB) = \#AB/\#\Omega,$$

and

$$P(B|A) = \frac{\#AB/\#\Omega}{\#A/\#\Omega} = \frac{\#AB}{\#A}.$$

*Note:* The sample space has been reduced.

## Example

A family is known to have two children, at least one of whom is a girl. What is the probability that the other is a boy?

Let

$$\Omega = \{bb, bg, gb, gg\},$$

$$A = \{bg, gb, gg\},$$

$$B = \{bb, bg, gb\},$$

Then

$$P(B|A) = \frac{\#AB}{\#A} = \frac{2}{3}$$

## Three Formulas

If  $P(A) > 0$ , then

$$P(AB) = P(A)P(B|A). \quad (1)$$

If  $0 < P(A) < 1$ , then  $B = AB \cup A^cB$ . So,

$$P(B) = P(AB) + P(A^cB),$$

and

$$P(B) = P(A)P(B|A) + P(A^c)P(B|A^c). \quad (2)$$

Finally,

$$P(A|B) = \frac{P(AB)}{P(B)}.$$

So,

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}. \quad (3)$$

## Example

Diagnostic Tests

**The Story.** A diagnostic test for a rare disease (e.g. an Xray for lung cancer) is part of a routine physical exam.

$D = \{\text{Disease is present}\}$ ,

$E = \{\text{Test Indicates disease present}\}$ .

Suppose

$$P(D) = .001,$$

$$P(E|D) = .98,$$

$$P(E|D^c) = .01.$$

Then

$$P(E) = P(D)P(E|D) + P(D^c)P(E|D^c).$$

So,

$$P(E) = .001 \times .98 + .999 \times .01 = .01097.$$

## Diagnostic Tests

Continued

Next

$$P(D|E) = \frac{P(D)P(E|D)}{P(D)P(E|D) + P(D^c)P(E|D^c)}.$$

So,

$$P(D|E) = \frac{.00098}{.01097} = .0893,$$

and

$$P(D|E) < .09.$$

*Note:*  $P(D)$  increased by a factor of (roughly 900), but is still small.

*Note:*

$$\begin{aligned}P(E|D^c) &= .01 \\P(E^c|D) &= 1 - P(E|D) = .02\end{aligned}$$

are called the *false negative* and *false positive rates*.

**Ref:** *Matters of Life and Death*, by Eugene Robin, M.D.

## Sampling Experiments

Sample 2 from  $R$  (e.g. 3) red tickets and  $N - R$  (e.g. 3) white tickets. Let

$$A = \{\text{red on first draw}\},$$

$$B = \{\text{red on second}\}.$$

**w.o.r..** For sampling *w.o.r.*,

$$P(A) = \frac{R(N-1)}{N(N-1)} = \frac{R}{N},$$

$$P(AB) = \frac{R(R-1)}{N(N-1)},$$

$$P(B|A) = \frac{R-1}{N-1}.$$

and, similarly,

$$P(B|A^c) = \frac{R}{N-1}.$$

So,

$$\begin{aligned}P(B) &= P(A)P(B|A) + P(A^c)P(B|A^c) \\&= \frac{R}{N} \frac{R-1}{N-1} + \frac{N-R}{N} \frac{R}{N-1} \\&= \frac{R^2 - R + NR - R^2}{N(N-1)} \\&= \frac{R(N-1)}{N(N-1)} \\&= \frac{R}{N}\end{aligned}$$

and

$$\begin{aligned}P(A|B) &= \frac{P(AB)}{P(B)} \\&= \frac{R(R-1)/N(N-1)}{R/N} \\&= \frac{R-1}{N-1}.\end{aligned}$$

*Notes a).* Proportional to # red balls.

*b).* Symmetry.

**w.r..** For sampling *w.r.*,

$$P(A) = \frac{RN}{N^2} = \frac{R}{N},$$

$$P(B) = \frac{NR}{N^2} = \frac{R}{N},$$

$$P(AB) = \frac{R^2}{N^2},$$

$$P(B|A) = \frac{R}{N}.$$

## Extensions

**The Product Rule:** If  $A_1, \dots, A_m$  are events for which

$$P(A_1 \cdots A_{m-1}) > 0,$$

then

$$P(A_1 \cdots A_m) = P(A_1) \prod_{k=2}^m P(A_k | A_1 \cdots A_{k-1}).$$

For, the right side is

$$P(A_1) \frac{P(A_1 A_2)}{P(A_1)} \frac{P(A_1 A_2 A_3)}{P(A_1 A_2)} \cdots \frac{P(A_1 \cdots A_m)}{P(A_1 \cdots A_{m-1})};$$

and cancellation leaves only  $P(A_1 \cdots A_m)$ .

**Example:** *The Birthday Problem.* Let  $A_k$  be the event that the  $k^{\text{th}}$  birthday differs from the first  $k-1$ . Then  $P(A_1) = 1$ ,

$$P(A_k | A_1, \dots, A_{k-1}) = \frac{365 - k + 1}{365},$$

$$P(A_1 \cdots A_n) = \prod_{k=1}^n \frac{365 - k + 1}{365}.$$

**The Partition Rule.** Let  $A_1, \dots, A_m$  be mutually exclusive and exhaustive; that is

$$A_i A_j = \emptyset \text{ for } i \neq j,$$

$$A_1 \cup A_2 \cup \cdots \cup A_m = \Omega.$$

Then

$$P(B) = \sum_{i=1}^m P(A_i B) = \sum_{i=1}^m P(A_i) P(B | A_i)$$

for any event  $B$

**Bayes Theorem.** If  $P(B) > 0$ , then

$$P(A_j | B) = \frac{P(A_j B)}{P(B)};$$

so,

$$P(A_j | B) = \frac{P(A_j) P(B | A_j)}{\sum_{i=1}^m P(A_i) P(B | A_i)}.$$

## Example

### Gold Coins

**The Story:**

Box 1 contains two silver coins.

Box 2 contains one gold and one silver coin.

Box 3 contains two gold coins.

Select a box (at random).

Examine the two coins in order.

*Note:* All choices are equally likely at each stage.

Let

$$A_k = \{\text{Box } k \text{ is selected}\},$$

$$B = \{\text{First coin is gold}\},$$

$$C = \{\text{Second coin is gold}\},$$

Then

$$P(A_k) = \frac{1}{3}$$

$$P(B | A_k) = \begin{cases} 0 & \text{if } k = 1 \\ 1/2 & \text{if } k = 2 \\ 1 & \text{if } k = 3 \end{cases}$$

$$P(BC | A_k) = \begin{cases} 0 & \text{if } k = 1 \text{ or } 2 \\ 1 & \text{if } k = 3 \end{cases}$$

So,

$$P(B) = \left(\frac{1}{3}\right) \times 0 + \left(\frac{1}{3}\right) \times \left(\frac{1}{2}\right) + \left(\frac{1}{3}\right) \times 1 \\ = \frac{1}{2}$$

$$P(BC) = \left(\frac{1}{3}\right) \times 0 + \left(\frac{1}{3}\right) \times 0 + \left(\frac{1}{3}\right) \times 1 \\ = \frac{1}{3}$$

and

$$P(C | B) = \frac{P(BC)}{P(B)} = \frac{1/3}{1/2} = \frac{2}{3}.$$

## A Theorem

If  $P(A) > 0$ , then

$$P(\Omega|A) = 1, \\ 0 \leq P(B|A) \leq 1,$$

for all events  $B$ . If  $B_1, B_2, \dots$  are mutually exclusive, then

$$P\left(\bigcup_{k=1}^{\infty} B_k|A\right) = \sum_{k=1}^{\infty} P(B_k|A).$$

*Proof*—See the text.

**Corollary:**  $P(B^c|A) = 1 - P(B|A)$ .

**Example:** Sampling *w.o.r.* with  $R = 2$  and  $N - R = 4$ . There

$$P(B|A) = \frac{1}{5}, \\ P(B|A^c) = \frac{2}{5},$$

and

$$P(B|A) + P(B|A^c) \neq 1.$$

## Independence

Events  $A$  and  $B$  are *independent* iff

$$P(AB) = P(A)P(B).$$

Note: If also  $P(A) > 0$ , then

$$P(B|A) = P(B).$$

*Examples: Sampling.*  $A$  and  $B$  were independent for sampling *w.r.*, but not for sampling *w.o.r.*.

*Example: Cards.* If a card is selected from a standard deck, let

$$A = \{\text{ace}\}, \\ B = \{\text{spade}\}.$$

Then

$$P(A) = \frac{4}{52} = \frac{1}{13}, \\ P(B) = \frac{13}{52} = \frac{1}{4}, \\ P(AB) = \frac{1}{52} = P(A)P(B).$$

## Independence and Complements

**Theorem.** If  $A$  and  $B$  are independent, then so are  $A^c$  and  $B$ . For if  $A$  and  $B$  are independent, then

$$P(B) = P(AB) + P(A^cB) \\ = P(A)P(B) + P(A^cB)$$

and, therefore,

$$P(A^cB) = P(B) - P(A)P(B) \\ = P(B)[1 - P(A)] \\ = P(A^c)P(B).$$

**Corollary:**  $A$  and  $B^c$  are independent, as are  $A^c$  and  $B^c$ .

## Several Events

Events  $A_1, \dots, A_n$  are *pairwise independent* iff

$$P(A_i A_j) = P(A_i)P(A_j)$$

for all  $1 \leq i < j \leq n$ ;  $A_1, \dots, A_n$  are *mutually independent* iff

$$P(A_i A_j) = P(A_i)P(A_j) \text{ for } i < j, \\ P(A_i A_j A_k) = P(A_i)P(A_j)P(A_k) \text{ for } i < j < k,$$

$\dots$ ,

$$P(A_{i_1} \cdots A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k}),$$

for all  $1 \leq i_1 < \dots < i_k \leq n$  and  $k = 2, \dots, n$ .

*Notes* a). Symmetry and Subsets.

b). Mutual implies pairwise.

c) "Independence" means "mutual independence."

### Example

#### Sampling With Replacement

If a sample  $n$  is drawn from  $R$  red and  $N - R$  white tickets *w.r.*. Let

$$A_k = \{\text{red on the } k^{\text{th}} \text{ draw}\}.$$

Then

$$P(A_j) = \frac{R}{N}$$

and

$$P(A_{i_1} \cdots A_{i_k}) = \frac{R^k N^{n-k}}{N^n} = \left(\frac{R}{N}\right)^k$$

for all  $1 \leq i_1 < \cdots < i_k \leq n$  and  $k = 2, \dots, n$ . So,  $A_1, \dots, A_n$  are mutually independent.

### Example

Draw one card from a standard deck. Let

$$A = \{\text{Spades or Clubs}\},$$

$$B = \{\text{Hearts or Clubs}\},$$

$$C = \{\text{Diamonds or Clubs}\}.$$

Then

$$P(A) = \frac{26}{52} = \frac{1}{2},$$

$$P(B) = P(C) = \frac{1}{2}.$$

Next

$$P(AB) = P(\{\text{Clubs}\}) = \frac{13}{52} = \frac{1}{4},$$

$$P(AC) = P(BC) = P(\{\text{Clubs}\}) = \frac{1}{4}.$$

So,  $A$ ,  $B$ , and  $C$  are pairwise independent.

However,

$$P(ABC) = P(\{\text{Clubs}\}) = \frac{1}{4}.$$

So,  $A$ ,  $B$ , and  $C$  are not mutually independent.

### Independence and Complements

**Theorem.**  $A_1, \dots, A_m$  are mutually independent iff

$$P(B_1 \cdots B_m) = P(B_1) \cdots P(B_m)$$

whenever

$$B_i = A_i \text{ or } A_i^c$$

for  $i = 1, \dots, m$ .

*Proof-Outline.* "Only if:" Combine earlier theorem with induction.

"If:" For example,

$$\begin{aligned} P(A_1 A_2) &= P(A_1 A_2 A_3) + P(A_1 A_2 A_3^c) \\ &= P(A_1)P(A_2)P(A_3) \\ &\quad + P(A_1)P(A_2)P(A_3^c) \\ &= P(A_1)P(A_2)[P(A_3) + P(A_3^c)] \\ &= P(A_1)P(A_2). \end{aligned}$$

### Example

#### Series and Parallel Connections

**The Story:** Given  $n$  electrical connections, let

$$A_k = \{\text{\(k^{\text{th}}\) one works}\};$$

and suppose that  $A_1, \dots, A_n$  are independent.

**Series Connections.** The probability that current flows is

$$P(A_1 \cdots A_n) = P(A_1) \times \cdots \times P(A_n).$$

**Parallel Connections:** The probability that current flows is

$$P(A_1 \cup \dots \cup A_n) = 1 - P(A_1^c \dots A_n^c).$$

Here

$$P(A_1^c \dots A_n^c) = P(A_1^c) \times \dots \times P(A_n^c)$$

So,

$$P(A_1 \cup \dots \cup A_n) = 1 - \prod_{k=1}^n [1 - P(A_k)].$$