Continuous Random Variables

Chapter 5
Important Concepts

- Distribution Functions.
- Density Functions.
- The Median and Other Quantiles.
- The Mean and Variance
- Special Distributions
  - Uniform
  - Exponential
  - Normal

Continuous Random Variables

Introductory Example
Random Angles

For the Uniform Spinner,

$$\Omega = (-\pi, \pi],$$

$$P(a, b] = \frac{b - a}{2\pi}$$

for $-\pi \le a < b \le \pi$. Let

$$X(\omega) = \omega$$

for $-\pi < \omega \le \pi$. Then X is random variable.

Note: $P[X = x] = 0$ for all $x \in \mathbb{R}$.

Distributions Functions

Def. Given a model (Ω, P) and a RV

$$X : \Omega \to \mathbb{R},$$

the *distribution function of* X is defined by

$$F(x) = P[X \le x] \qquad (*)$$

for $-\infty < x < \infty$.

Example. For the random angle

$$F(x) = \frac{1}{2} + \frac{x}{2\pi}$$

for $-\pi < x \le \pi$, since

$$\begin{aligned} P[X \le x] &= P[-\pi < X \le x] \\ &= P((\pi, x]) = \frac{x + \pi}{2\pi}. \end{aligned}$$

Notation: Write $X \sim F$ if X is a RV with DF F; that is, if (*) holds.

Two Important Relations

If $X \sim F$, then

$$P[a < X \le b] = F(b) - F(a)$$

for all $-\infty < a < b < \infty$, since

$$\begin{aligned} F(b) = P[X \le b] &= P[X \le a] + P[a < X \le b] \\ &= F(a) + P[a < X \le b] \end{aligned}$$

Also,

$$P[X > b] = 1 - P[X \le b] = 1 - F(b).$$

Example. In the random angle example,

$$F(x) = \frac{1}{2} + \frac{x}{2\pi},$$

$$\begin{aligned} F(\tfrac{2}{3}\pi) - F(\tfrac{1}{3}\pi) &= (\frac{1}{2} + \frac{1}{3}) - (\frac{1}{2} + \frac{1}{6}) \\ &= \frac{1}{6}. \end{aligned}$$

Some Definitions

Def. A function $F : I\!\!R \to I\!\!R$ is *non-decreasing* if $F(x) \le F(y)$ whenver $x \le y$.

One Sided Limits

$$F(x+) = \lim_{y \to x, y > x} F(y),$$

and

$$F(x-) = \lim_{y \to x, y < x} F(y).$$

Characteristic Properties

Theorem. A function $F : I\!\!R \to I\!\!R$ is the distribution function of some RV iff

$$F(x) \le F(y) \text{ whenever } x \le y, \tag{1}$$

$$F(x) = \lim_{y \downarrow x} F(y) \text{ each all } x, \tag{2}$$

$$\lim_{x \to -\infty} F(x) = 0, \tag{3a}$$

$$\lim_{x \to \infty} F(x) = 1. \tag{3b}$$

Proof. Later.

Note: Henceforth DF means a function satisfying (1), (2), and (3).

The Discrete Case

If X is discrete with range $\mathcal{X}$ and PMF f, then

$$\begin{aligned} F(x) &= P[X \le x] \\ &= \sum_{y \in \mathcal{X}, y \le x} f(y). \end{aligned}$$

If also

$$\mathcal{X} \subseteq \{\cdots - 1, 0, 1, 2, \cdots\},$$

then

$$F(n) = \sum_{k \le n} f(k)$$

and

$$f(n) = F(n) - F(n-1).$$

Densities

Def. A function $f : I\!\!R \to I\!\!R$ is a *density* if

$$f(x) \ge 0, \text{ for all } x,$$
$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

If $X \sim F$ and f is a density then X *and* F *have density* f if

$$F(x) = \int_{-\infty}^{x} f(y)dy \tag{4}$$

for all $-\infty < x < \infty$.

Theorem. If f is any density, then (4) defines a DF.

Proof. Exercise.

Corollary. If f is any density, then there is a RV X with density f.

Consequences Of

$$F(x) = \int_{-\infty}^{x} f(y)dy \tag{4}$$

If $X \sim F$ with density f, then

$$P[a < X \leq b] = F(b) - F(a) = \int_{a}^{b} f(x)dx$$

for $-\infty \leq a < b \leq \infty$.

If (4) holds, then

$$f(x) = F'(x) = \frac{d}{dx}F(x)$$

at continuity points of f.

Example

Uniform Distributions

If $-\infty < \alpha < \beta < \infty$, then

$$f(x) = \begin{cases} 1/(\beta - \alpha) & if\ \alpha < x \leq \beta \\ 0 & if otherwise \end{cases}$$

is a density, since $f(x) \geq 0$ for all x and

$$\begin{aligned} \int_{\infty}^{\infty} f(x)dx &= \int_{\alpha}^{\beta} \frac{dx}{\beta - \alpha} \\ &= \frac{1}{\beta - \alpha}(\beta - \alpha) = 1. \end{aligned}$$

Then

$$F(x) = \begin{cases} 0 & if\ x \leq \alpha \\ (x-\alpha)/(\beta - \alpha) & if\ \alpha < x \leq \beta \\ 1 & if\ x > \beta \end{cases}$$

Example: *Random Angles*: $\alpha = -\pi$ and $\beta = \pi$.

Standard $\alpha = 0$ and $\beta = 1$.

Example

Exponential Distributions

If $0 < \lambda < \infty$, then

$$F(x) = \begin{cases} 0 & if\ x \leq 0 \\ 1 - e^{-\lambda x} & if else \end{cases}$$

has density

$$f(x) = \begin{cases} 0 & if\ -\infty < x < 0 \\ \lambda e^{-\lambda x} & if\ 0 \leq x < \infty. \end{cases}$$

Notes a). Derivative doesn't exist when $x = 0$.

b). Doesn't matter.

The Standard Normal Density

Fact

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}dz = \sqrt{2\pi}.$$

Let

$$\varphi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$$

Then φ is a density, called *the standard normal density*. The *standard normal* distribution function is

$$\Phi(z) = \int_{-\infty}^{z} \varphi(y)dy.$$

Note: Tabled

Note:

$$\begin{aligned} \Phi(-z) &= \int_{-\infty}^{-z} \varphi(y)dy \\ &= \int_{z}^{\infty} \varphi(x)dx = 1 - \Phi(z) \end{aligned}$$

and

$$\Phi(0) = \frac{1}{2}.$$

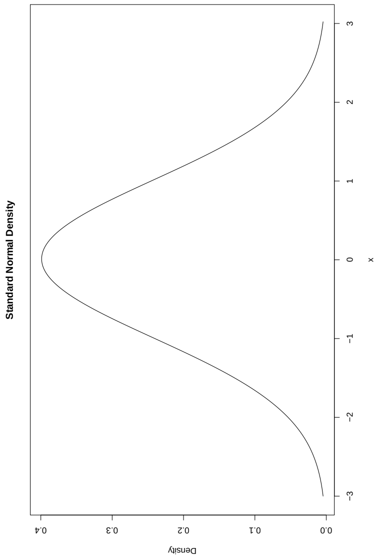

Figure 1: The Standard Normal Density

The Normal Distribution With Parameters

Let

$$f(x) = \frac{1}{\sigma}\varphi\left(\frac{x-\mu}{\sigma}\right)$$

and

$$F(x) = \int_{-\infty}^{x} f(y)dy = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

These are the normal distribution with parameters $-\infty < \mu < \infty$ and $\sigma > 0$.

The Median and Other Quantiles

If $X \sim F$, then and m for which

$$F(m) = \frac{1}{2}$$

is called a *median* of X or F. More generally, if $0 < p < 1$, then any x for which

$$F(x) = p$$

is called a p^{th}*-quantile or* $100p^{th}$ *percentile* of X or F. In terms of X, the condition is

$$P[X \leq x] = p.$$

Example

If

$$F(t) = \begin{cases} 0 & if\ \infty < t < 0 \\ 1 - e^{-\lambda t} & if\ 0 \leq t < \infty \end{cases}$$

then

$$F(m) = \frac{1}{2}$$

iff

$$1 - e^{-\lambda m} = \frac{1}{2}$$

iff

$$e^{-\lambda m} = \frac{1}{2}$$

iff

$$e^{\lambda m} = 2$$

iff

$$m = \frac{1}{\lambda}\log(2),$$

where log denotes natural logarthim.

Example: The half life of a radio active substance.

The Mean and Variance

Def. If X has density f, then the *mean of X and/or f* is defined by

$$\mu = \int_{-\infty}^{\infty} x f(x) dx,$$

provided that the integral converges absolutely.

Example: *Uniform Distributions*. If

$$f(x) = \begin{cases} 1/(\beta - \alpha) & if \alpha < x \leq \beta \\ 0 & if otherwise \end{cases}$$

then

$$\begin{aligned} \mu &= \int_{\alpha}^{\beta} \frac{xdx}{\beta - \alpha} \\ &= \frac{1}{2} \frac{x^2}{\beta - \alpha} |_{x=\alpha}^{\beta} \\ &= \frac{1}{2} \frac{\beta^2 - \alpha^2}{\beta - \alpha} \\ &= \frac{\alpha + \beta}{2}, \end{aligned}$$

since $\beta^2 - \alpha^2 = (\beta + \alpha)(\beta - \alpha)$.

Variance And Standard Deviation

The variance of X and/or f is

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx.$$

Also,

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) - \mu^2$$

The *standard deviation* is

$$\sigma = \sqrt{\sigma^2}.$$

An Example

A Uniform Distribution

If

$$f(x) = \begin{cases} 1 & if\ 0 \leq x \leq 1 \\ 0 & if\ otherwise \end{cases}$$

then

$$\mu = \int_0^1 xdx = \frac{1}{2} x^2 |_{x=0}^{1} = \frac{1}{2},$$

$$\int_0^1 x^2 dx = \frac{1}{3} x^3 |_{x=0}^{1} = \frac{1}{3}$$

and

$$\sigma^2 = \int_0^1 x^2 dx - \mu^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

More generally, if

$$X \sim \text{Unif}(\alpha, \beta),$$

then

$$\sigma^2 = \frac{(\beta - \alpha)^2}{12}.$$

Normal Distribution Function

The Median and Mean The mean and median of a normal distribution function

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

are both μ, since

$$F(\mu) = \Phi(0) = \frac{1}{2},$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} x f(x) dx &= \mu + \int_{-\infty}^{\infty} (x - \mu) f(x) dx \\ &= \mu + \sigma \int_{-\infty}^{\infty} z \varphi(z) dz = \mu. \end{aligned}$$

The Variance Similarly, the variance is σ^2.

The Gamma Function

Let

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

for $\alpha > 0$. Then

$$\Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x}|_{x=0}^{\infty} = 1,$$

and

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$$

for $\alpha > 0$. For, integrating by parts,

$$\begin{aligned}\Gamma(\alpha) &= \int_0^\infty x^{\alpha} e^{-x} dx \\ &= -x^{\alpha} e^{-x}|_{x=0}^{\infty} + \int_0^\infty \alpha x^{\alpha-1} e^{-x} dx \\ &= \alpha\Gamma(\alpha).\end{aligned}$$

So

$$\Gamma(n) = (n-1)!.$$

The Mean and Variance Of Exponential Distributions

If

$$f(x) = \begin{cases} 0 & if x < 0 \\ \lambda e^{-\lambda x} & if\ 0 \le x < \infty \end{cases}$$

then

$$\begin{aligned}\mu &= \int_0^\infty x\lambda e^{-\lambda x} dx \\ &= \frac{1}{\lambda}\int_0^\infty x e^{-\lambda x} dx \\ &= \frac{1}{\lambda}\Gamma(2) \\ &= \frac{1}{\lambda}\end{aligned}$$

The Mean and Variance Of Exponential Distributions

Continued

Similarly,

$$\begin{aligned}&\int_0^\infty x^2 \lambda e^{-\lambda x} dx \\ &\qquad = \frac{1}{\lambda^2}\int_0^{\infty^2} x^2 e^{-\lambda x} dx \\ &\qquad = \frac{1}{\lambda^2}\Gamma(3) \\ &\qquad = \frac{2}{\lambda^2}\end{aligned}$$

and

$$\sigma^2 = 2\frac{}{\lambda^2} - \mu^2 = \frac{1}{\lambda^2}.$$