Random Variables

Chapter 4

Important Concepts

- Random variables.
- Probability mass functions.
- The mean and variance.
- Special Distributions

  Hypergeometric

  Binomial

  Negative Binomial

  Poisson

Random Variables

Consider a probability model (Ω, P).

Definition. A *random variable* is a function

$$X : \Omega \to \mathbb{R}.$$

If X is a RV, let

$$\mathcal{X} = \{X(\omega) : \omega \in \Omega\},$$

the range of X. Then X is *discrete* if

$$\mathcal{X} = \{x_1, x_2, \cdots\}.$$

(finte or infinite).

Examples

Coin Tossing: If a penny and a nickel are tossed, then

$$\Omega = \{hH, hT, tH, tT\},$$
$$P(A) = \frac{\#A}{4}.$$

Define $X : \Omega \to \mathbb{R}$, by

$$X(tT) = 0,$$
$$X(tH) = X(hT) = 1,$$
$$X(hH) = 2.$$

Then

$$P[X = 0] = P(\{tT\}) = 1/4,$$
$$P[X = 1] = P(\{tH, hT\}) = 1/2,$$
$$P[X = 2] = P(\{hH\}) = 1/4,$$

Indicators: Let $A \subseteq \Omega$ and $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and 0 otherwise. Then $P[\mathbf{1}_A = 1] = P(A)$ and $P[\mathbf{1}_A = 0] = P(A^c)$.

Probability (Mass) Functions

Notation (just used). If X is a random variable write

$$P[X \in B] = P(\{\omega : X(\omega) \in B\})$$

for $B \subseteq \mathbb{R}$.

Definition. If X is a discrete RV, then the *probability mass function of* X is defined by

$$f(x) = P[X = x].$$

for $x \in \mathbb{R}$.

Example. In the coin tossing example

$$\mathcal{X} = \{0, 1, 2\},$$
$$f(0) = 1/4,$$
$$f(1) = 1/2,$$
$$f(2) = 1/4,$$

and $f(x) = 0$ for other x.

Alternative Notation: f_X.

Hypergeometric Distributions

If a sample of n is drawn $w.o.r.$ from $R \geq 1$ red and $N - R \geq 1$ white, then the PMF of

$$X = \#\text{red tickets in the sample}$$

is

$$f(r) = \binom{R}{r}\binom{N-R}{n-r}/\binom{N}{n}$$

for $r = 0, \cdots, n$ and $f(x) = 0$ for other x.

Proof.

$$\#\Omega = \binom{N}{n}.$$

and

$$\#\{X = r\} = \binom{R}{r}\binom{N-R}{n-r}.$$

Def. Called *hypergeometric* with parameters R, N, and n.

Committees. With $R = 5$, $N = 10$, and $n = 4$.

Properties of PMFs

If f is the PMF of a RV X with range $\mathcal{X}$, then

$$f(x) \geq 0 \text{ for all } x, \tag{1}$$

$$f(x) = 0 \text{ unless } x \in \mathcal{X}, \tag{2}$$

$$\sum_{x \in \mathcal{X}} f(x) = 1, \tag{3}$$

and

$$P[X \in B] = \sum_{x \in B} f(x) \tag{4}$$

for $B \subseteq I\!R$.

Conversely, any function f that satisfies (1), (2), and (3) is the PMF of some RV X.

Notes *a)*. Henceforth "PMF" means any function that satisfies (1), (2), and (3).

b). We can specify a model by giving $\mathcal{X}$ and f, subject to the first three conditions.

The Proof

Only If

Suppose first that $f(x) = P[X = x]$. Then (1) and (2) are clear. If $B \in I\!R$, then $B \cap \mathcal{X} = \{x'_1, x'_2, \cdots\}$. So,

$$\begin{aligned} P[X \in B] &= P[X \in B \cap \mathcal{X}] \\ &= P[\bigcup_k \{X = x'_k\}] \\ &= \sum_k P[X = x'_k] \\ &= \sum_{x \in B} f(x). \end{aligned}$$

This is (4); (3) follows by setting $B = \mathcal{X}$, in which case the left side in 1.

The Proof

If

For the converse, let f and $\mathcal{X}$ be given. Then let $\Omega = \mathcal{X}$ and

$$P(A) = \sum_{x \in A} f(x),$$

$$X(\omega) = \omega.$$

Then, for all $x \in \mathcal{X}$,

$$P[X = x] = P(\{x\}) = f(x)$$

The Mean and Variance

Defs. If X has PMF f, then the *mean, variance*, and *standard deviation* of X are

$$\mu = \sum_{x \in X} x f(x)$$

$$\sigma^2 = \sum_{x \in \mathcal{X}} (x-\mu)^2 f(x).$$

and

$$\sigma = \sqrt{\sigma^2}.$$

Notes *a)* Depend only on f

b) Units.

Example: Committees

$$f(x) = \binom{5}{x}\binom{5}{4-x} \Big/ \binom{10}{4}.$$

x	$f(x)$	$xf(x)$	$(x-\mu)^2 f(x)$
0	$\frac{5}{210}$	0	$\frac{20}{210}$
1	$\frac{50}{210}$	$\frac{50}{210}$	$\frac{50}{210}$
2	$\frac{100}{210}$	$\frac{200}{210}$	$\frac{0}{210}$
3	$\frac{50}{210}$	$\frac{150}{210}$	$\frac{50}{210}$
4	$\frac{5}{210}$	$\frac{20}{210}$	$\frac{20}{210}$
Totals	1	2	$\frac{2}{3}$

So,

$$\mu = 2,$$
$$\sigma^2 = \frac{2}{3}.$$

Useful Formulas

If $X \sim f$, then,

$$\sigma^2 = \sum_{x \in \mathcal{X}} x^2 f(x) - \mu^2,$$

since

$$\begin{aligned}\sigma^2 &= \sum_{x \in \mathcal{X}} (x^2 - 2\mu x + \mu^2) f(x) \\ &= \sum_{x \in \mathcal{X}} x^2 f(x) - 2\mu \sum_{x \in \mathcal{X}} x f(x) + \mu^2 \sum_{x \in \mathcal{X}} f(x) \\ &= \sum_{x \in \mathcal{X}} x^2 f(x) - 2\mu^2 + \mu^2.\end{aligned}$$

Similarly, if $\mathcal{X} \subseteq \{0, 1, 2, \cdots\}$, then

$$\sigma^2 = \sum_{x \in \mathcal{X}} (x)_2 f(x) + \mu - \mu^2,$$

where $(x)_2 = x(x-1)$.

Independent Trials

A *basic experiment with* Ω_0 is repeated n times. Events depending on different trials (replications) are independent.

Example*a)*. Sampling with replacement.

b). Coin Tossing.

c). Roulette.

Basic Question. If $A_0 \subseteq \Omega_0$, what is the probability that A_0 occurs k times in n trials? Formally, letting

$$A_i = \{A_0 \text{ occurs on the } i^{th} \text{ trial}\},$$
$$X = \mathbf{1}_{A_1} + \cdots + \mathbf{1}_{A_n},$$

what is

$$P[X = k].$$

An Example

Roulette

$n = 4$, $k = 2$

Let

$$W_i = \{\text{Win on } i^{th} \text{ Game},$$
$$L_i = W_i^c = \{\text{Lose on } i^{th} \text{ Game},$$

Then

$$P(W_i) = \frac{9}{19} = p, \text{ say},$$
$$P(L_i) = 1 - p = \frac{10}{19} = q \text{ say}.$$

Let

$$X = \mathbf{1}_{W_1} + \mathbf{1}_{W_2} + \mathbf{1}_{W_3} + \mathbf{1}_{W_4},$$

$$\begin{aligned}\{X = 2\} = &W_1W_2L_3L_4 \cup W_1L_2W_3L_4 \\ &\cup W_1L_2L_3W_4 \cup L_1W_2W_3L_4 \\ &\cup L_1W_2L_3W_4 \cup L_1L_2W_3W_4.\end{aligned}$$

So, letting

$$\begin{aligned}P[X = 2] &= P(W_1W_2L_3L_4) \\ &+ \cdots \\ &+ P(L_1L_2W_1W_2) \\ &= P(W_1)P(W_2)P(L_3)P(L_4) \\ &+ \cdots \\ &+ P(L_1)P(L_2)P(W_1)P(W_2) \\ &= ppqq + pqpq + pqqp \\ &+ qppq + qpqp + qqpp \\ &= 6p^2q^2.\end{aligned}$$

Note: Numerically

$$P[X = 2] = 6(\frac{9}{19})^2(\frac{10}{19})^2 = .373.$$

Binomial Distributions

Let $A_1, \cdots, A_n$ be independent events for which

$$P(A_i) = p, \ i = 1, \cdots, n;$$

and let

$$X = \mathbf{1}_{A_1} + \cdots + \mathbf{1}_{A_n}.$$

Then

$$P[X = k] = \binom{n}{k} p^k q^{n-k}$$

for $k = 0, \cdots, n$, where $q = 1 - p$.

Why-Briefly. We may choose k trials in $\binom{n}{k}$ ways; and

$$\begin{aligned}P[X = k] &= \binom{n}{k} P(A_1 \cdots A_k A_{k+1}^c \cdots A_n^c) \\ &= \binom{n}{k} P(A_1) \cdots P(A_k) P(A_{k+1}^c) \cdots P(A_n^c) \\ &= \binom{n}{k} p^k q^{n-k}.\end{aligned}$$

Example

Q: What is the probability that South gets no aces on at least $k = 5$ or $n = 9$ hands?

A: Let

$$A_i = \{\text{no aces on the } i^{th} \text{hand}\},$$
$$X = \mathbf{1}_{A_1} + \cdots + \mathbf{1}_{A_9}.$$

Then

$$P(A_i) = .3038 = p, \text{ say}$$

for $i = 1, \cdots, 9$. So,

$$P[X = k] = \binom{9}{k} p^k (1 - p)^{n-k}$$

for $k = 0, \cdots, 9$, and

$$P[X \geq 5] = \sum_{k=5}^{9} \binom{9}{k} p^k (1 - p)^{n-k}.$$

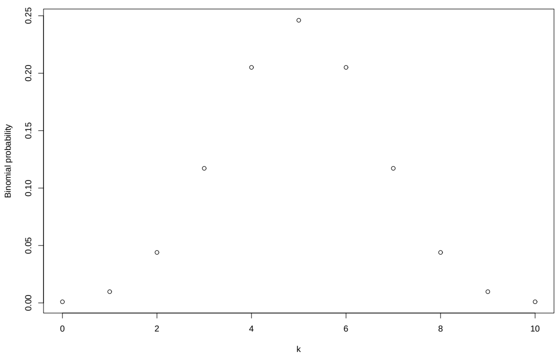
So,

$$P[X \geq 5] = .1035.$$

Graph Of f(k)

With $n = 10$ and $p = .5$

$$f(k) = \binom{10}{k}(\frac{1}{2})^{10}$$



Binomial Distributions

Def. The function

$$f(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for $k = 0, \cdots, n$ and $f(x) = 0$ otherwise, is called the *Binomial PMF with parameters n and p.*

The Mean and Variance:

$$\mu = np,$$
$$\sigma^2 = npq.$$

The Proof: The Mean

Let $q = 1 - p$. Then

$$\mu = \sum_{k=0}^{n} k f(k) = \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k}.$$

Here

$$k\binom{n}{k} = k\frac{(n)_k}{k!}$$
$$= n\frac{(n-1)_{k-1}}{(k-1)!} = n\binom{n-1}{k-1}.$$

So,

$$\mu = \sum_{k=1}^{n} n\binom{n-1}{k-1} p^k q^{n-k}$$
$$= np\sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} q^{(n-1)-(k-1)}$$
$$= np,$$

since the sum is the sum of binomial probabilities with parameters $n - 1$ and p.

Inverse Sampling

An Example

A company must hire three engineers. Each interview results in a hire with probability

$$p = \frac{1}{3}.$$

What is the probability that 10 interviews are required. We need

- Two hires on the first nine interviews.
- Success on the 10^{th}.

The probability of both is

$$\binom{9}{2}(\frac{1}{3})^2(\frac{2}{3})^7 \times (\frac{1}{3}) = \binom{9}{2}(\frac{1}{3})^3(\frac{2}{3})^7.$$

Inverse Sampling
Negative Binomial Distributions

Let $A_1, A_2, \cdots$ be independent with $P(A_i) = p$ and

$$X_n = \mathbf{1}_{A_1} + \cdots + \mathbf{1}_{A_n},$$

$$\begin{aligned} Y_1 &= \text{smallest } n \text{ with } X_n \geq 1, \\ Y_2 &= \text{smallest } n \text{ with } X_n \geq 2, \\ &\cdots, \\ Y_r &= \text{smallest } n \text{ with } X_n \geq r, \end{aligned}$$

Then

$$\begin{aligned} P[Y_r = n] &= P[\{X_{n-1} = r-1\} \cap A_n] \\ &= P[X_{n-1} = r-1]P(A_n) \\ &= \binom{n-1}{r-1} p^{r-1} q^{n-r} \times p \\ &= \binom{n-1}{r-1} p^r q^{n-r} \end{aligned}$$

for $n = r, r+1, \cdots$.

Negative Binomial Distributions

See Text for Detaila

Let

$$f(n) = \binom{n-1}{r-1} p^r q^{n-r}$$

for $n = r, r+1, \cdots$ and $f(x) = 0$ for other values of x.Then

$$\sum_{n=r}^{\infty} f(n) = 1.$$

So, f is a PMF, called the *Negative Binomial PMF with parameters r and p.*

The Mean and Variance: These are

$$\begin{aligned} \mu &= \frac{r}{p}, \\ \sigma^2 &= \frac{rq}{p^2}. \end{aligned}$$

Expression For e^x

$e = 2.7183 \cdots$

First Expression:

$$e^x = \lim_{n \to \infty} \big(1 + \frac{x}{n}\big)^n$$

Second Expression

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

Poisson Distributions

AKA The Law of Rare Events

The Derivation: Consider the binomial with

$$\begin{aligned} n &\to \infty, \\ p &\to 0, \\ 0 < \lambda = np &< \infty \text{ constant.} \end{aligned}$$

Then

$$p = \frac{\lambda}{n}$$

and

$$\begin{aligned} \binom{n}{k} p^k (1-p)^{n-k} &= \frac{1}{k!} (n)_k \big(\frac{\lambda}{n}\big)^k \big(1 - \frac{\lambda}{n}\big)^{n-k} \\ &= \frac{1}{k!} \lambda^k \big(\frac{(n)_k}{n^k}\big) \big(1 - \frac{\lambda}{n}\big)^{n-k}. \end{aligned}$$

Here

$$\lim \frac{(n)_k}{n^k} = 1, \lim \big(1 - \frac{\lambda}{n}\big)^{n-k} = e^{-\lambda}.$$

for each fixed k.

So,

$$\lim \binom{n}{k} p^k (1-p)^{n-k} = \frac{1}{k!}\lambda^k e^{-\lambda}$$

Let

$$f(k) = \frac{1}{k!}\lambda^k e^{-\lambda}$$

for $k = 0, 1, 2, \cdots$ and $f(x) = 0$ for other values of x. Then f is a PMF, since

$$\sum_{k=0}^{\infty} f(k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{1}{k!}\lambda^k = e^{-\lambda} e^{\lambda} = 1.$$

The Poisson PMF: f is called the *Poisson PMF with parameter λ.*

The Mean and Variance: These are

$$\mu = \lambda,$$
$$\sigma^2 = \lambda.$$

Example

Q: A professor hits the wrong key with probability $p = .001$ each time he types a letter (or symbols). What is the probability of 5 or more errors in $n = 2500$ letters.

A: Let X be the number of errors. Then is is approximately Poisson with

$$\lambda = 2500 \times .001 = 2.5.$$

So,

$$P[X = k] = \frac{1}{k!}(2.5)^k e^{-2.5}$$

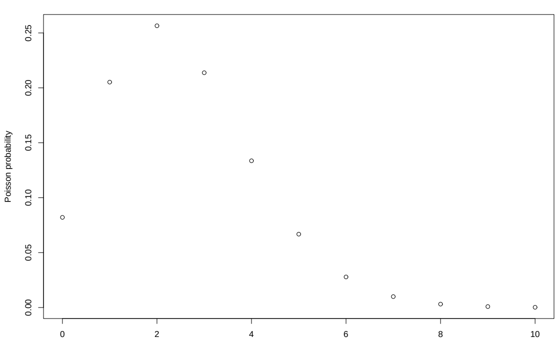
for $k = 0, 1, 2, \cdots$ and

$$\begin{aligned} P[X \geq 5] &= 1 - P[X \leq 4] \\ &= 1 - \sum_{k=0}^{4} \frac{1}{k!}(2.5)^k e^{-2.5} \\ &= .1088 \end{aligned}$$

Graph Of f(k)

With $\lambda = 2.5$

$$f(k) = \frac{1}{k!}(2.5)^k e^{-2.5}$$



The Mean and Variance of the Poisson

The Derivation

Here

$$\begin{aligned} \mu &= \sum_{k=0}^{\infty} k \times \frac{\lambda^k}{k!} e^{-\lambda} \\ &= \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} \\ &= \lambda. \end{aligned}$$

Similarly,

$$\sum_{k=0}^{\infty} k^2 \times \frac{\lambda^k}{k!} e^{-\lambda} = \lambda + \lambda^2.$$

So,

$$\sigma^2 = (\lambda + \lambda^2) - \mu^2 = \lambda.$$