Multiple Integrals

If

$$-\infty < a < b < \infty,$$
$$-\infty < c < d < \infty,$$

and

$$f : [a, b] \times [c, d] \to I\!R,$$

is sufficiently nice (e.g. continuous), then

$$\int_a^b \int_c^d f(x, y) dy dx$$

is the limit of Riemann sums.

Extentions: Under mild technical conditions,

$$a, c \to -\infty,$$
$$b, d \to \infty.$$

Other Regions: Under conditions,

$$\int \int_C f(x, y) dy dx$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \mathbf{1}_C(x, y) dy dx.$$

where

$$\mathbf{1}_C(x, y) = \begin{cases} 1 & if \ (x, y) \in C \\ 0 & if \ otherwise. \end{cases}$$

Special Case:

$$\int \int_C 1 dy dx = \text{Area}(C).$$

Example

$$\int \int_{x^2+y^2 \le r^2} dx dy = \pi r^2.$$

Reduction to Iterated Integration

If

$$C = \{(x, y) : a \le x \le b, \ c(x) \le y \le d(x)\},$$

then

$$\int \int_C f(x, y) dy dx = \int_a^b \big[\int_{c(x)}^{d(x)} f(x, y) dy \big] dx,$$

under mild conditions.

Reduction to Iterated Integration Continued

Recall

$$\int \int_{a \le x \le b, c(x) \le y \le d(x)} f(x, y) dy dx$$
$$= \int_a^b \left[\int_{c(x)}^{d(x)} f(x, y) dy \right] dx,$$

Example: If $T = \{(x, y) : 0 \le x \le 1, \ 0 \le y \le x\}$, then

$$\int \int_T 1 dy dx = \int_0^1 \big[\int_0^x dy \big] dx$$
$$= \int_0^1 x dx = \frac{1}{2}.$$

Consequence

$$\int_a^b \int_c^d g(x) h(y) dy dx = \big[\int_a^b g(x) dx \big] \big[\int_c^d h(y) dy \big].$$

Higher Dimensions

Write

$$\int_{\mathbb{R}^m} f(\mathbf{x})d\mathbf{x}$$

for

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_1,\cdots,x_m)dx_m\cdots dx_1.$$

Then

Bivariate Densities

A function

$$f:\mathbb{R}^2\to\mathbb{R}$$

is a *bivariate density* if

$$f(x,y)\ge 0,$$

and

$$\int\int_{R^2} f(x,y)dydx = 1.$$

If f is a density, then JDRVs X and Y have *joint density* f

$$P[(X,Y)\in C] = \int\int_C f(x,y)dydx$$

for nice subsets $C\subseteq\mathbb{R}^2$.

Example: Uniform Distributions. If $R\subset\mathbb{R}^2$ and $0<\alpha=\text{Area}(R)<\infty$, then

$$f(x,y)=\frac{1}{\alpha}\mathbf{1}_R(x,y)$$

is a density, called *the uniform density over* R.

Marginal Densities

If X and Y have joint density f, then X and Y have individual (marginal) densities

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)dy,$$
$$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)dx.$$

Example: Let

$$D=\{(x,y): x^2+y^2\le 1\}$$

and

$$f(x,y)=\frac{1}{\pi}\mathbf{1}_D(x,y).$$

If $-1<x<1$, then

$$f_X(x)=\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{1}{\pi}dy=\frac{2}{\pi}\sqrt{1-x^2}.$$

Example

If

$$f(x,y)=\frac{2}{(1+x+y)^3}$$

for $0\le x,y<\infty$ and $f(x,y)=0$ otherwise, then

$$\begin{aligned}\int_0^{\infty} f(x,y)dy &= -\frac{1}{(1+x+y)^2}\Big|_{y=0}^{\infty}\\ &= \frac{1}{(1+x)^2}\end{aligned}$$

for $0\le x<\infty$, and

$$\begin{aligned}\int\int_{\mathbb{R}^2} f(x,y)dydx &= \int_0^{\infty}\frac{dx}{(1+x)^2}\\ &= -\frac{1}{1+x}\Big|_{x=0}^{\infty}\\ &= 1.\end{aligned}$$

So, f is a density and

$$f_X(x)=\frac{1}{(1+x)^2}$$

for $0\le x<\infty$.

Multivariate Densities A function

$$f : \mathbb{R}^m \to \mathbb{R}$$

is a *m-variate density* if

$$f(\mathbf{x}) \geq 0,$$

and

$$\int_{R^m} f(\mathbf{x}) d\mathbf{x} = 1.$$

If f is a density, then JDRVs $X_1, \cdots, X_m$ have *joint density* f

$$P[\mathbf{X} \in C] = \int_C f(\mathbf{x}) d\mathbf{x}$$

for nice subsets $C \subseteq \mathbb{R}^m$.

Marginal Densities: If $X_1, \cdots, X_j$ and $Y_1, \cdots, Y_k$ have joint density f, then $X_1, \cdots, X_j$ have joint density

$$f_{\mathbf{X}}(\mathbf{x}) = \int_{\mathbb{R}^k} f(\mathbf{x}, \mathbf{y}) d\mathbf{y}.$$

Joint Distribution Functions

Def: If X and Y are JDRVs, then their *joint distribution function* is

$$F(a, b) = P[X \leq a, Y \leq b].$$

Marginal Distributions: Then

$$F_X(a) = \lim_{b \to \infty} F(a, b),$$
$$F_Y(b) = \lim_{a \to \infty} F(a, b),$$

Notes a) Characteristic Properties

b) Higher Dimensions

c) Harder to Use

d) Mixed Distributions

Independence

JDRVs X and Y are *independent* if

$$P[X \in A, \; Y \in B] = P[X \in A] P[Y \in B]$$

for all nice subsets $A, B \subseteq \mathbb{R}$ (for example, intervals).

Conditions for Independence

DFs: X and Y are independent iff

$$F(a, b) = F_X(a) F_Y(b)$$

for all $a, b \in \mathbb{R}$.

Densities: If X and Y have individual densities f_X and f_Y, then X and Y are independent iff X and Y have joint density

$$f(x, y) = f_X(x) f_Y(y).$$

Example

If

$$X \sim \text{Exp}(\lambda),$$
$$Y \sim \text{Exp}(\lambda)$$

are independent, what is

$$P[Y \geq 2X \text{ or } X \geq 2Y].$$

Here

$$f_X(z) = f_Y(z) = \lambda e^{-\lambda z}$$

for $0 \leq z < \infty$. So,

$$f(x, y) = f_X(x) f_Y(y) = \lambda^2 e^{-\lambda(x+y)}$$

for $0 \leq x, y < \infty$ and $f(x, y) = 0$ for other x and y. So,

$$\begin{aligned} P[Y \geq 2X] &= \int_0^\infty \Big[\int_{2x}^\infty \lambda^2 e^{-\lambda(x+y)} dy\Big] dx \\ &= \int_0^\infty -\lambda e^{-\lambda(x+y)}\big|_{y=2x}^\infty dx \\ &= \int_0^\infty \lambda e^{-3\lambda x} dx \\ &= -\frac{1}{3} e^{-3\lambda x}\big|_{x=0}^\infty \\ &= \frac{1}{3}. \end{aligned}$$

Similarly,

$$P[X \geq 2Y] = \frac{1}{3}.$$

So,

$$P[Y \geq 2X \text{ or } X \geq 2Y] = \frac{2}{3}.$$

Several Variables

$X_1, \cdots, X_m$ are independent if

$$P[X_1 \in A_1, \cdots, X_m \in A_m] = P[X_1 \in A_1] \times \cdots \times P[X_m \in A_m]$$

Note: Equivalent Conditions. For example,

$$f(x_1, \cdots, x_m) = f_1(x_1) \times \cdots \times f_m(x_m).$$

The Distribution of the Maximum

Example: n light globes are placed in service at time $t = 0$ and allowed to burn continuously. Denote their lifetimes by $X_1, \cdots, X_n$ and suppose that

$$X_1, \cdots, X_n \sim^{ind} F. \qquad (*)$$

If burned out globes are not replaced, then the room goes dark at time

$$Y = \max[X_1, \cdots, X_n],$$

the largest of $X_1, \cdots, X_n$.

The Distribution of Y: If (*) holds, then

$$\begin{aligned} F_Y(y) &:= P[Y \leq y] \\ &= P[X_1 \leq y, \cdots, X_n \leq y] \\ &= P[X_1 \leq y] \times \cdots \times P[X_n \leq y] \\ &= F(y) \times \cdots \times F(y) \\ &= F(y)^n. \end{aligned}$$

So, if F has density f, then Y has density

$$F_Y(y) = \frac{d}{dy} F(y)^n = nF(y)^{n-1} f(y).$$

Example: Revisited. If $n = 5$ and F is exponential with $\lambda = 1$ per mo., then $F(t) = 1 - e^{-t}$,

$$F_Y(t) = (1 - e^{-t})^5,$$

and

$$f_Y(t) = 5(1 - e^{-t})^4 e^{-t}$$

for $0 \leq t < \infty$. The probability that the room is still lighted after two months is

$$\begin{aligned} P[Y > 2] &= 1 - F_Y(2) \\ &= 1 - (1 - e^{-2})^5 \\ &= .5167. \end{aligned}$$

Order Statistics

If

$$X_1, \cdots, X_n \sim^{ind} F,$$

let

$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$$

by $X_1, \cdots, X_n$ in increasing order. Thus,

$$X_{(1)} = \min[X_1, \cdots, X_n],$$
$$\cdots,$$
$$X_{(n)} = \max[X_1, \cdots, X_n].$$

Notes

- Times that globes burn out–in the example.
- Can find distributions.
- Section 6.6 and Problems 9 and 10.

Convolutions

The Continuous Case

Let X and Y are independent with densities f_X and f_Y, and let

$$Z = X + Y.$$

Then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx.$$

Example: If $X \sim \text{Unif}[0,1]$ and $Y \sim \text{Unif}[0,1]$, then

$$f_Z(z) = \min[z, 2-z]$$

for $0 \leq z \leq 2$ and $f_Z(z) = 0$ otherwise.

In this case

$$f_X(z) = f_Y(z) = 1 \text{ for} 0 \leq z \leq 1,$$
$$f_X(z) = f_Y(z) = 0 \text{ otherwise.}$$

So, if $0 \leq z \leq 1$, for example, then

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx \\ &= \int_0^z 1 \times 1 dx \\ &= z. \end{aligned}$$

Similarly

Let

$$X_1, \cdots, X_n \text{ be independent}$$

and

$$Y = X_1 + \cdots + X_n.$$

If

$$X_i \sim \text{Gamma}(\alpha_i, \beta),\ i = 1, \cdots, n,$$

then

$$Y \sim \text{Gamma}(\alpha_1 + \cdots + \alpha_n, \beta).$$

If

$$X_i \sim \text{Normal}(\mu_i, \sigma_i^2),\ i = 1, \cdots, n,$$

then

$$Y \sim \text{Normal}(\mu, \sigma^2),$$

where

$$\mu = \mu_1 + \cdots + \mu_n,$$
$$\sigma^2 = \sigma_1^2 + \cdots + \sigma_n^2.$$

Conditional Distributions

The Continuous Case

Conditional Densities: Let X and Y have joint density f. If $f_X(x) > 0$, then the *conditional conditional of Y given X* is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}.$$

As above, this is a density.

Conditional Probability: Write

$$P[Y \in B | X = x] = \int_B f_{Y|X}(y|x) dy.$$

Notes: • New definition.

- P[X = x] = 0.
- Can reverse the roles of X and Y.

Example

If

$$f(x,y) = \frac{2}{(1+x+y)^3}$$

for $0 \le x, y < \infty$, then

$$f_X(x) = \frac{1}{(1+x)^2}$$

for $0 \le x < \infty$. So,

$$f_{Y|X}(y|x) = \frac{2(1+x)^2}{(1+x+y)^3}$$

and

$$\begin{aligned} P[Y > c|X = x] &= \int_c^\infty \frac{2(1+x)^2}{(1+x+y)^3} dy \\ &= -\frac{(1+x)^2}{(1+x+y)^2}\Big|_{y=c}^\infty \\ &= \frac{(1+x)^2}{(1+x+c)^2}. \end{aligned}$$

Bayes Theorem

In both cases (discrete and continuous),

$$f(x,y) = f_{Y|X}(y|x) f_X(x),$$

if $f_X(x) > 0$. In the discrete case,

$$f_Y(y) = \sum_{x \in \mathcal{X}} f(x,y) \qquad (*)$$

and

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)},$$

when $f_Y(y) > 0$. In the continuous case, the sum in (*) is replaced by an integral.

Mixed Distributions: One variable can be discrete and the other continous.

Multivariate Extensions: X and/or Y can be vectors.

The Rule of Succession

Suppose

$$X \sim \text{Unif}[0,1],$$

and

$$\begin{aligned} &f_Y(y_1, \cdots, y_n|x) \\ &\qquad = x^{y_1+\cdots+y_n}(1-x)^{n-(y_1+\cdots+y_n)} \end{aligned}$$

for $y_1, \cdots, y_n = 0$ or 1. Then

$$\begin{aligned} P[Y_1 = 1, \cdots, Y_n = 1] &= f_Y(1, \cdots, 1) \\ &= \int_0^1 x^n dy \\ &= \frac{1}{n+1}. \end{aligned}$$

If there were an $(n+1)^{st}$ Y, then

$$\begin{aligned} P[Y_{n+1} = 1 | Y_1 = 1, \cdots, Y_n = 1] \\ &= \frac{P[Y_1 = 1, \cdots, Y_{n+1} = 1]}{P[Y_1 = 1, \cdots, Y_n = 1]} \\ &= \frac{1/(n+2)}{1/(n+1)} \\ &= \frac{n+1}{n+2}. \end{aligned}$$

Note: Depends (crucially) on the distribution of X.

Transformations

Let

$$m \geq 1,$$
$$D \subseteq \mathbb{R}^m,$$

and let

$$w : D \to^{onto} E \subseteq \mathbb{R}^m$$

be surjective (one-to-one). Write

$$\mathbf{y} = w(\mathbf{x})$$

as

$$\begin{aligned} y_1 &= w_1(x_1, \cdots, x_m), \\ y_2 &= w_2(x_1, \cdots, x_m), \\ &\cdots, \\ y_m &= w_m(x_1, \cdots, x_m), \end{aligned}$$

Jacobians

Assuming that w is differentiable, let

$$J_w(\mathbf{x}) = |\det \begin{pmatrix} \frac{\partial w_1(\mathbf{x})}{\partial x_1} & \cdots & \frac{\partial w_1(\mathbf{x})}{\partial x_m} \\ & \cdots & \\ & \cdots & \\ \frac{\partial w_m(\mathbf{x})}{\partial x_1} & \cdots & \frac{\partial w_m(\mathbf{x})}{\partial x_m} \end{pmatrix}|,$$

where

$$\mathbf{x} = (x_1, \cdots, x_m).$$

Or, briefly,

$$J(\mathbf{x}) = |\det \frac{\partial w(\mathbf{x})}{\partial \mathbf{x}}|.$$

Polar Coordinates

Let

$$m = 2,$$
$$D = (0, \infty) \times [-\pi, \pi),$$

$$y_1 = r\cos(\theta),$$
$$y_2 = r\sin(\theta).$$

Then

$$w : D \to^{onto} \mathbb{R}^2 - \{0\}$$

$$\frac{\partial w_1}{\partial r} = \cos(\theta),$$
$$\frac{\partial w_2}{\partial r} = \sin(\theta),$$
$$\frac{\partial w_1}{\partial \theta} = -r\sin(\theta),$$
$$\frac{\partial w_2}{\partial \theta} = r\sin(\theta),$$

So,

$$J(r,\theta) = |\det \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{pmatrix}|$$

That is,

$$\begin{aligned} J(r,\theta) &= r\cos^2(\theta) + r\sin^2(\theta) \\ &= r. \end{aligned}$$

Change of Variables

Suppose that

$$w : D \to^{onto} E$$

is differentiable and one-to-one and let $v : E \to D$ be the inverse function.

Theorem. Under techinical conditions,

$$\int_D g(\mathbf{x})d\mathbf{x} = \int_E g[v(\mathbf{y})]J_v(\mathbf{y})d\mathbf{y}.$$

Corollary.

$$\begin{aligned} &\int\int_{\mathbb{R}^2} g(x_1,x_2)dx_2dx_1 \\ &\qquad = \int_0^\infty \int_{-\pi}^{\pi} g[r\cos(\theta), r\sin(\theta)]rd\theta dr. \end{aligned}$$

Transformations of RV's

Now let

$$X_1, \cdots, X_m \sim f \text{ joint density.}$$

Let

$$\mathbf{X} = (X_1, \cdots, X_m).$$

Suppose

$$P[X \in D] = 1$$

and let

$$\mathbf{Y} = w(\mathbf{X}),$$

where

$$w : D \to^{onto} E \subseteq \mathbb{R}^m.$$

That is,

$$\begin{aligned} Y_1 &= w_1(X_1, \cdots, X_m), \\ Y_2 &= w_2(X_1, \cdots, X_m), \\ &\cdots, \\ Y_m &= w_m(X_1, \cdots, X_m), \end{aligned}$$

Suppose that

$$w \text{ is surjective,}$$
$$v = w^{-1} \text{ is smooth.}$$

Theorem. $\mathbf{Y}$ has density

$$g(\mathbf{y}) = f[v(\mathbf{y})]J_v(\mathbf{y})\mathbf{1}_E(\mathbf{y}).$$

Corollary

$$g_1(y_1) = \int_{\mathbb{R}^{m-1}} g(y_1, \mathbf{z})d\mathbf{z}.$$

Corollary. If

$$\begin{aligned} X_1, X_2 &\sim f, \\ X_1 &= R\cos(\Theta), \\ X_2 &= R\sin(\Theta), \end{aligned}$$

then

$$g(r,\theta) = f[r\cos(\theta), r\sin(\theta)]r$$

for $0 < r < \infty$ and $-\pi < \theta \le \pi$.

Example

If

$$X_1, X_2 \sim^{ind} \Phi,$$

then

$$\begin{aligned} f(x_1, x_2) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_1^2} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_2^2} \\ &= \frac{1}{2\pi} e^{-\frac{1}{2}(x_1^2 + x_2^2)}. \end{aligned}$$

$$\begin{aligned} X_1 &= R\cos(\Theta), \\ X_2 &= R\sin(\Theta), \end{aligned}$$

then

$$\begin{aligned} g(r, \theta) &= f[r\cos(\theta), r\sin(\theta)]r \\ &= \frac{r}{2\pi} e^{-\frac{1}{2}r^2} \end{aligned}$$

for $0 < r < \infty$ and $-\pi < \theta \leq \pi$.

So,

$$\begin{aligned} g_1(r) &= \int_{-\pi}^{\pi} \frac{r}{2\pi} e^{-\frac{1}{2}r^2} d\theta \\ &= re^{-\frac{1}{2}r^2} \end{aligned}$$

for $0 < r < \infty$; and

$$\begin{aligned} g_2(\theta) &= \int_0^{\infty} \frac{r}{2\pi} e^{-\frac{1}{2}r^2} dr \\ &= -\frac{1}{2\pi} e^{-\frac{1}{2}r^2} |_{r=0}^{\infty} \\ &= re^{-\frac{1}{2}r^2}; \end{aligned}$$

and

$$R \quad \text{and} \quad \Theta \qquad \text{are independent.}$$