Limit Theorems

Chapter 8

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Markov's Inequality

If Y is any RV and $0 < c < \infty$, then

$$P[|Y| \geq c] \leq \frac{1}{c} E|Y|.$$

Proof. Let

$$B = \{|Y| \geq c\}.$$

Then

$$c\mathbf{1}_B \leq |Y|.$$

So,

$$E|Y| \geq E(c\mathbf{1}_B) = cP(B).$$

Chebyshev's Inequality

If X has mean and variance

$$\mu = E(X),$$
$$\sigma^2 = E[(X-\mu)^2],$$

then

$$P[|X-\mu| \geq c] \leq \frac{\sigma^2}{c^2}.$$

Proof. Let

$$Y = |X-\mu|^2$$

in Markov's Inequality. Then

$$\begin{aligned} P[|X-\mu| \geq c] &= P[Y \geq c^2] \\ &\leq \frac{1}{c^2} E|Y| \\ &= \frac{\sigma^2}{c^2}. \end{aligned}$$

Alternative Statement: Let

$$X^* = \frac{X-\mu}{\sigma}.$$

Then

$$E(X^*) = 0,$$
$$D^2(X^*) = 1,$$

and

$$P[|X^*| \geq c] \leq \frac{1}{c^2}.$$

Remark: General, but seldom sharp.

Example: If $X \sim \text{Normal}[\mu, \sigma^2]$, then $X^* \sim \Phi$, and

$$P[|X^*| \geq 2] = \cdots = .046,$$

from the normal tables. Chebyshev asserts (only)

$$P[|X^*| \geq 2] \leq \frac{1}{4}.$$

Sums of Independent RVs

If

$$
\begin{aligned}
X_1, \cdots, X_n &\text{ are independent,} \\
E(X_i) &= \mu_i, \\
D^2(X_i) &= \sigma_i^2
\end{aligned}
$$

and

$$S = X_1 + \cdots + X_n,$$

then

$$
\begin{aligned}
E(S) &= \mu_1 + \cdots + \mu_n, \\
D^2(S) &= \sigma_1^2 + \cdots + \sigma_n^2.
\end{aligned}
$$

Special Case: If $\mu_i = \mu$ and $\sigma_i^2 = \sigma^2$ for $i = 1, \cdots, n$, then

$$
\begin{aligned}
E(S) &= n\mu, \\
D^2(S) &= n\sigma^2, \\
D(S_n) &= \sigma\sqrt{n}.
\end{aligned}
$$

Special Cases: Continued. Let

$$\bar{X} = \frac{S}{n} = \frac{X_1 + \cdots + X_n}{n}.$$

Then

$$E(\bar{X}) = \frac{1}{n}E(S) = \frac{1}{n}n\mu = \mu,$$

and

$$
\begin{aligned}
D^2(\bar{X}) &= (\frac{1}{n})^2 D^2(S) \\
&= (\frac{1}{n})^2 n\sigma^2 \\
&= \frac{\sigma^2}{n}.
\end{aligned}
$$

Note: $D^2(\bar{X}) \to 0$ as $n \to \infty$. So, for any $\epsilon > 0$,

$$
\begin{aligned}
P[|\bar{X} - \mu| \geq \epsilon] &\leq \frac{1}{\epsilon^2} D^2(\bar{X}) \\
&= \frac{\sigma^2}{n\epsilon^2} \\
&\to 0
\end{aligned}
$$

as $n \to \infty$.

An Example

If

$$X_1, \cdots, X_n \text{ are independent}$$

and

$$X_i = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } q = 1 - p \end{cases},$$

then

$$
\begin{aligned}
E(X_i) &= p, \\
D^2(X_i) &= p(1-p) \leq \frac{1}{4},
\end{aligned}
$$

and, therefore,

$$
\begin{aligned}
E(\bar{X}) &= p, \\
D^2(\bar{X}) &= \frac{p(1-p)}{n} \leq \frac{1}{4n}.
\end{aligned}
$$

Question: Using Chebyshev, find an N for which

$$P[|\bar{X} - p| \geq .01] \leq .01$$

for all $n \geq N$. Here

$$
\begin{aligned}
P[|\bar{X} - p| \geq .01] &\leq \frac{p(1-p)}{n(.01)^2} \\
&\leq \frac{2500}{n} \\
&\leq .01
\end{aligned}
$$

iff

$$n \geq \frac{2500}{.01} = 250,000.$$

Note: Conservative estimate.

Type of Convergence

If Y_n are RVs and c is a constant, then Y_n *converges to c in mean square* iff

$$\lim_{n\to\infty} E[(Y_n - c)^2] = 0;$$

and Y_n *converges to c in probability* iff

$$\lim_{n\to\infty} P[|Y_n - c| \geq \epsilon] = 0$$

for all $\epsilon > 0$.

A Simple Relation: If $Y_n \to^{ms} c$, then $Y_n \to^{p} c$, since

$$\begin{aligned} P[|Y_n - c| \geq \epsilon] &= P[|Y_n - c|^2 \geq \epsilon^2] \\ &\leq \frac{1}{\epsilon^2} E|Y_n - c|^2, \end{aligned}$$

by Markov's Inequality.

The Law of Large Numbers

Theorem. Let F be a distribution function with mean and variance

$$\mu = \int_{-\infty}^{\infty} x dF(x),$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 dF(x);$$

and let

$$X_1, \cdots, X_n \sim^{ind} F.$$

Then

$$\bar{X}_n = \frac{X_1 + \cdots + X_n}{n} \to \mu$$

in mean square and in probability.

Proof. Here

$$E[(\bar{X}_n - \mu)^2] = D^2(\bar{X}_n) = \frac{\sigma^2}{n} \to 0,$$

as $n \to \infty$.

Paraphase: For repeated trials. Let

$$\text{Time Ave } = \frac{X_1 + \cdots + X_n}{n}$$

and

$$\text{Space Ave } = \mu = \int_{-\infty}^{\infty} x dF(x).$$

Then

$$\text{Time Ave } = \text{Space Ave.}$$

Note: Can predict long run behavior.

Example

Roulette

A Single Game: The expected gain is

$$X = \begin{cases} 1 & \text{w.p. } 9/19 \\ -1 & \text{w.p. } 10/19 \end{cases},$$

then

$$E(X) = \frac{9}{19} - \frac{10}{19} = -\frac{1}{19}.$$

Many Games: The actual average game per play is

$$\frac{X_1 + \cdots + X_n}{n} \to -\frac{1}{19}.$$

Note: Qualitative result; quantified later.

Indicators

If $A_1, A_2, \cdots$ are independent with

$$P(A_i) = p,$$

then

$$E(\mathbf{1}_{A_i}) = p,$$

and

$$\frac{1}{n}[\mathbf{1}_{A_1} + \cdots + \mathbf{1}_{A_n}] \to p.$$

Note: This is the frequentist interpretation of "probability."

The Law of Averages: Let

$$N_n = \mathbf{1}_{A_1} + \cdots + \mathbf{1}_{A_n}.$$

Then

$$P[A_{n+1}|N_n = k] = p$$

for all k (assuming that p is fixed).

The Central Limit Theorem

Recall: If F is a DF with mean μ and variance σ^2,

$$X_1, \cdots, X_n \sim^{ind} F,$$
$$S_n = X_1 + \cdots + X_n,$$

then

$$E(S_n) = n\mu,$$
$$D^2(S_n) = n\sigma^2.$$

Let

$$S_n^* = \frac{S_n - E(S_n)}{D(S_n)} = \frac{S_n - n\mu}{\sigma\sqrt{n}}.$$

Also, let

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}y^2} dy.$$

The Central Limit Theorem. For all real z,

$$\lim_{n\to\infty} P[S_n^* \le z] = \Phi(z). \qquad (*)$$

That is

$$P[S_n^* \le z] \approx \Phi(z)$$

for large n. Equivalently,

$$P[S_n \le x] \approx \Phi(\frac{x - n\mu}{\sigma\sqrt{n}}).$$

Remarks *a).* (*) is true for *any* F.

b). Speed of convergence does depend on F.

c). Importance of the normal distribution.

Approximations

Example

Let

$$T = \text{Tax},$$
$$\langle T\rangle = \text{closest integer},$$
$$X = T - \langle T\rangle.$$

Suppose

$$X \sim \text{Unif}(-\frac{1}{2}, \frac{1}{2}).$$

Then

$$\mu = E(X) = 0,$$
$$\sigma^2 = Var(X) = \frac{1}{12}.$$

Let

$$n = 12,000,000,$$
$$X_1, \cdots, X_n \sim^{ind} \text{Unif}(-\frac{1}{2}, \frac{1}{2}),$$
$$S = X_1 + \cdots + X_n.$$

Question: How big is S?

Worst Case Analysis: $|S| \leq 6,000,000$, but this is very conservative.

Probabilistic Analysis: By CLThm,

$$S \approx \text{Normal}[n\mu = 0, n\sigma^2 = 1,000,000].$$

Note:

$$\sqrt{n\sigma^2} = 1000.$$

So,

$$\begin{aligned} P[-2000 \leq S \leq 2000] & \\ &\approx \Phi(\frac{2000-0}{1000}) - \Phi(\frac{-2000-0}{1000}) \\ &= \Phi(2) - \Phi(-2) \\ &= .954. \end{aligned}$$