Statistics 425
Practice Final

Note: For the final examination, you may use a calculator and *one* page of note (two-sided).

1. Let S denote the total sum of spots when a balanced die is rolled $n = 100$ times. Approximate the probability that the total sum of spots is between 325 and 375 (inclusive). Find a c for which $P[350 - c \le X \le 350 + c] \approx .95$.

2. Describe a simulation experiment that could be used to compute the probabilty that $XY \le 1/2$ when X and Y are independent random variables that are uniformly distributed over $(0, 1]$. The description should include an error bound.

Ans: Generate $2N$ uniformly distributed random number, $X_1, Y_1, \cdots, X_N, Y_N$; count the number of times $XY \le 1/2$; and divide by N. The error bound is $1/\sqrt{N}$.

3. Let X have the bilateral exponential distribution with density $f(x) = \frac{1}{2}e^{-|x|}$ for $-\infty < x < \infty$. Find the moment generating function of X and use it to compute the mean and variance of X.

Ans: $M(t) = 1/(1 - t^2)$ for $-1 < t < 1$; $\mu = 0$ and $\sigma^2 = 2$.

4. Now let $X_1, \cdots, X_{100}$ be independent random variable, all of which have the bilateral exponential distribution. Use Chebyshev's and Bernstein's/Chernoff's Inequalities to bound $P[|\bar{X}| \ge .5]$. (The Bernstein/Chernoff Inequality is called the "Bernstein's Inequality" in the on line notes and "Chernoff's Inequality" in the text.)

Ans: .08 and .0046

5. Let X_1, X_2, X_3 be independent standard normal random variables, and let $Y_1 = X_1 + X_2$ and $Y_2 = X_2 + X_3$. Find the mean and variance of Y_1 and the covariance betweeen Y_1 and Y_2. Also, find the best linear predictor of Y_2, using Y_1.

Ans: 0, 2, and 1; $Y_1/2$.

6. Find the joint density of Y_1 and Y_2 in the previous problem.

Hint: Y_1 and Y_2 have a bivariate normal distribution.

7. Suppose that a simple random sample of size five is drawn without replacement from a box containing five each of red tickets, white tickets, blue tickets, green tickets, and black tickets. Find the expected number of different colors in the sample.

Ans: 3.54

8. In the previous problem, suppose that the sample size is ten. Find the probability that all colors are represented in the sample.

Ans: .7265.

9. If X and Y are independent random variables that are uniformly distributed over $(-\pi, \pi]$, what is the mean of $Z = X\cos(XY)$.

Ans: 0

10. A pair of balanced six-sided dice are rolled $n = 12$ times. Let X denote the number of rolls on which doubles appear, and let Y denote the number of rolls for which the sum of spots in equal to seven. Find the joint probability mass function of X and Y and compute $P[X = Y]$. What are the marginal probability mass functions of X and Y (individually).

Ans: $P[X = Y] = .200$; X and Y both have marginal binomial distributions with $n = 12$ and $p = 1/6$.

11. Suppose that X and Y have joint density $f(x, y) = x + y$ for $0 \leq x, y \leq 1$ and $f(x, y) = 0$ otherwise. Find $P[Y \leq \frac{1}{2} | X = \frac{1}{2}]$. Are X and Y independent?

Ans: $P[Y \leq \frac{1}{2} | X = \frac{1}{2}] = 3/8$; X and Y are not independent.

12. Tickets are drawn sequentially and at random from a box that contains 5 red and 5 white tickets. Let X denote the number of the draw on which the first red ticket appears. Find the probability mass function of X and compute the probability that X is odd.

Ans: $f(x) = 5 \times (5)_{x-1}/(10)_x$ for $x = 1, \cdots, 6$; .659

13. There are 730 students in a large Statistics class. Find the probability that at least three of them have birthdays on January 1.

Ans: .3233

14. Find the probability that a bridge hand (sample of 13 from a standard deck) is balanced (contains four cards of one suit and three of each of the others). Find the conditional probability that the hand is balanced, given that it contains four spades.

Ans: .105 and .110

15. Suppose that the probability that a family will have n children is 2^{-n-1} for $n \geq 0$ and that all children are equally likely to be female or male, independently of other children. What is the conditional probability that a family has at least one child, given that it has no boys?

Ans: 1/4.