Math-Stat 425
Problem Set 4
Due November 26

Instructions. Do all problems. Write out the solutions to any five of the even numbered problem. Show your work and explain your reasoning carefully.

1. In the game of bridge, an ace counts for 4 points, a king counts for 3, a queen for 2, and a jack for 1. Let Y denote number of points in a bridge hand. Find the mean of Y, and explain your reasoning.
Ans: 10.

2. n people put their names in a box, and then each person draws one name at random (without replacement). Let Y denote the number of people who draw their own names. Find the mean and variance of Y and explain your reasoning.

3. Let X and Y denote independent Poisson random variables both with mean $\lambda = 1$. Find the mean and variance of the product $Z = XY$.
Ans: 1 and 3.

4. Let $X_1, \cdots, X_n \sim^{ind} \text{Exp}(\lambda)$, and let $Y = \min[X_1, \cdots, X_n]$. Find the mean and variance of Y.

5. If $X \sim \text{Unif}(-\pi, \pi]$, what are the mean and variance of $Y = \sin(X)$.
Ans: 0 and 1/2.

6. A merchant buys 50 donuts each moring for 25 cents apiece and sells donuts for 50 cents apiece. Daily demand for donuts is a random variable, Y say, which is uniformly distributed over $1, \cdots, 100$, and unsold donuts are given away. Let X denote the merchant's daily profit. Find $E(X)$.

7. From past experience, a students scores on the midterm and final examination, X and Y say, are jointly distributed random variables with means $E(X) = E(Y) = 85$, standard deviations $\sigma_X = \sigma_Y = 10$, and correlation $\rho = .6$. Find the best linear predictor of the final exam score for student who scored 95 on the midterm.
Ans: 91

8. A signal Y is normally distributed with mean $E(Y) = 10$ and variance $\sigma_Y^2 = 5$. The signal is obscured by an independent normally distributed error Z with mean $E(Z) = 0$ and variance $\sigma_Z^2 = 1$, so that only $X = Y + Z$ is observed. Find the best linear predictor of Y based on X and the minimum mean squared error. How would your answers change if Y and $Z + 1$ had gamma distributions instead of normal ones?

9. Let $X, Y \sim^{ind} \text{Exp}(\lambda = 1)$, and let $W = X + Y$ and $Z = Y - X$. Find the conditional mean and variance of Z given $W = w$ for $w > 0$.
Ans: 0 and $w^2/3$.

10. In the previous problem, find the conditional mean and variance of W given $Z = z$ for arbitrary $-\infty < z < \infty$.

11. If X and Y are independent random variables with moment generating functions

$$E(e^{tX}) = \frac{e^t}{2-e^t},$$
$$E(e^{tY}) = \cosh(t),$$

find $P[X+Y=0]$.
Ans: 1/4.

12. Let X and Y be independent standard exponential random variables ($\lambda = 1$) and let $Z = Y - X$. Use moment generating functions to find the first four moments of Z. Then find the variance of Z^2.

13. Let X and Y be independent random variables with moment generating functions

$$E(e^{tX}) = e^{\frac{1}{2}t^2},$$
$$E(e^{tY}) = \frac{\sinh(t)}{t},\ t \neq 0.$$

Express the density of $X+Y$ in terms of the standard normal distribution function.
Ans: $[\Phi(x+1) - \Phi(x-1)]/2$.

Additional Suggested Problems, Chapter 7. Problems 5, 9, 25, 40, 45, 55,56, 77; Theoretical Exercises: 2, 7.