MATH/STAT 425
PROBLEM SET 2

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Note: For each problem, just one possible solution is provided. Your solutions can be different.

1. Solution to Question 2:

Define the following events
Q- Queen has hemophilia
NQ- Queen does not have hemophilia
NF- First Prince does not have homophilia
S- Second Prince has hemophiia
From problem 1 (3.70 in book) we have the following.
$P(Q) = 0.5$, $\quad P(S|Q) = 0.5$

Using this information we can now compute the probability the second Prince has hemophilia. Note that it is assumed that the diagnosis of the first son is independent (conditionally independent on the Queen that is) of the diagnosis of the second son.

$$\begin{aligned} P(S|NF) = \frac{P(NF \cap S)}{P(NF)} &= \frac{P(S \cap NF|Q)P(Q) + P(S \cap NF|NQ)P(NQ)}{P(NF|Q)P(Q) + P(NF|NQ)P(NQ)} \\ &= \frac{P(S|Q)P(NF|Q)P(Q) + P(S|NQ)P(NF|NQ)P(NQ)}{P(NF|Q)P(Q) + P(NF|NQ)P(NQ)} \\ &= \frac{(0.5)(0.5)(0.5) + (1)(0)(0.5)}{(0.5)(0.5) + (0.5)(1)} \\ &= \frac{1}{6} \end{aligned}$$

Assumption: If the Queen does not carry the gene, her son will not carry the gene (ie $P(S|NQ) = 0$)

2. Solution to Question 4:

Define the following events:
E_1- event there are no spades
E_2- event there are no hearts
E_3- event there are no clubs

E_4- event there are no diamonds

$$P(E_i) = \frac{\binom{39}{7}}{\binom{52}{7}} \qquad i = 1, 2, 3, 4$$

$$P(E_i E_j) = \frac{\binom{26}{7}}{\binom{52}{7}} \qquad i \neq j$$

$$P(E_i E_j E_k) = \frac{\binom{13}{7}}{\binom{52}{7}} \qquad i \neq j \neq k$$

$$P(E_i E_j E_k E_l) = 0$$

Using Proposition 4.4 in Chapter 2 of the book we have

$$\begin{aligned} P(\text{All suits are present}) &= 1 - P(\bigcup_{i=1}^{4} E_i) \\ &= 1 - [\binom{4}{1} P(E_i) - \binom{4}{2} P(E_i E_j) + \binom{4}{3} P(E_i E_j E_k) - \binom{4}{4} P(E_i E_j E_k E_l)] \\ &= 0.569578 \end{aligned}$$

3. Solution to Question 6:

We need to form a group of size four and we know that our group contains 1 woman and 1 man are in the group. There are many ways to do this problem but one way is to reduce the sample space to events which consist of a committee of size four with atleast one man and one woman.

$$\begin{aligned} P(2M, 2W|1M, 1W) &= \frac{P(2M, 2W)}{P(2M, 2W) + P(3M, 1W) + P(1M, 3W)} \\ &= \frac{\binom{10}{2}\binom{10}{2}}{\binom{10}{2}\binom{10}{2} + \binom{10}{1}\binom{10}{3} + \binom{10}{1}\binom{10}{3}} \\ &= \frac{27}{59} = 0.457627 \end{aligned}$$

4. Solution to Question 8:

Define the following events
K- event know answer
C- event you answer correctly
G- event guess answer
$P(K) = 0.7$
$P(G) = 0.3$
Assuming that when the answer is guessed it is a random choice and not based on

some intuition we have $P\left(C\,|G\right) = 0.5$

$$\begin{aligned} P(K|C) &= \frac{P(C|K)P(K)}{P(C|K)P(K) + P(C|G)P(G)} \\ &= \frac{(1)(0.7)}{(1)(0.7) + (0.5)(0.3)} \\ &= \frac{14}{17} = 0.8235294118 \end{aligned}$$

5. Solution to Question 10:

Define the follwoing events
C_i- the event that the i^{th} relay is closed.
S- current is succesfully flowing
Apply Proposition 4.4 of Chapter as follows.

$$\begin{aligned} P(S) = P(C_1C_4 \cup C_2C_5) \cup C_1C_3C_5 \cup C_2C_3C_4) \\ = P(C_1C_4) + P(C_2C_5) + P(C_1C_3C_5) + P(C_2C_3C_4) - P(C_1C_2C_4C_5) + \cdots \\ = 2(0.9_2) + 2(0.9^3) - 5(0.9^4) - 2(0.9^5) + 4(0.9^5) \\ = 0.97848 \end{aligned}$$

6. Solution to Question 12:

From question 11 we have that events A_1, A_2, ... are independent events with: $P(A_k) = \frac{1}{(1+k)^2}$ for each k and $B_n = \bigcup_{k=1}^n A_k$ and $B_\infty = \bigcup_{k=1}^\infty A_k$ for $n = 1, 2, \ldots$. Solving for $P\left(B_n\right)$ we get the following.

$$P(B_n) = 1 - P(B_n^c)$$

$$\begin{aligned} &= 1 - P(\bigcap_{n}^{k=1})A_k^c) \\ &= 1 - \prod_{k=1}^{n} P(A_k^c) \qquad \text{independence} \\ &= 1 - \prod_{k=1}^{n} (1 - \frac{1}{(k+1)^2}) \\ &= 1 - \prod_{k=1}^{n} \frac{k(k+2)}{(k+1)(k+1)} \\ &= 1 - \frac{(1)(3)}{(2)(2)} \frac{(2)(4)}{(3)(3)} \cdots \frac{(n-1)(n+1)}{(n)(n)} \frac{(n)(n+2)}{(n+1)(n+1)} \qquad \text{canceling terms} \\ &= 1 - (\frac{n+2}{2(n+1)}) \end{aligned}$$

Using the Monotone sequence theorem we see that our see our the sequence $a_n = 1 - \frac{n+2}{2(n+1)}$ is monotone increasing and bounded, so we may conclude that it has a finite limit which exists.
Taking this limit gives

$$P\left(B_{\infty}\right) = \lim_{n\to\infty} = 1 - \frac{n+2}{2(n+1)} = \frac{1}{2}.$$