Math/Stat 425
Problem Set 3

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Note: For each problem, just one possible solution is provided. You solutions can be different.

1 Solution to Question 2:

Denote X as the blood pressure. Thus $X \sim N(\mu, \sigma)$. Let $Z = (X - \mu)/\sigma$. Then $Z \sim N(0, 1)$.

The percentage of college age men having blood pressures between 115 and 135 is

$$\begin{aligned} P(115 \le X \le 135) &= P((115-\mu)/\sigma \le (X-\mu)/\sigma \le (135-\mu)/\sigma) \\ &= P((115-\mu)/\sigma \le Z \le (135-\mu)/\sigma) \\ &= P((115-120)/10 \le Z \le (135-120)/10) \\ &= P(-.5 \le Z \le 1.5) \\ &\approx 0.625. \end{aligned}$$

Similarly, the percentage having blood pressure over 135 is

$$\begin{aligned} P(X > 135) &= P((X-\mu)/\sigma > (135-\mu)/\sigma) \\ &= P(Z > (135-\mu)/\sigma) \\ &= P(Z > (135-120)/10) \\ &= P(Z > 1.5) \\ &\approx 0.067. \end{aligned}$$

2 Solution to Question 4:

Since $p(x,y)$ is a joint probability mass function,

$$\begin{aligned}
1 &= \sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty} p(x,y) \\
&= \sum_{x=1}^{\infty}\sum_{y=x}^{\infty} p(x,y) \\
&= \sum_{x=1}^{\infty}\sum_{y=x}^{\infty} c2^{-y} \\
&= c\sum_{x=1}^{\infty} 2\times 2^{-x} \\
&= 2c2\times 2^{-1} \\
&= 2c.
\end{aligned}$$

Thus $c = 1/2$.

When $x \in \mathbb{N}$, the marginal probability function of X is

$$\begin{aligned}
P(X=x) &= \sum_{y=x}^{\infty} P(X=x, Y=y) \\
&= \sum_{y=x}^{\infty} c\times 2^{-y} \\
&= 2^{-x}.
\end{aligned}$$

When $x \in \mathbb{Z}-\mathbb{N}$, the marginal probability function of X is

$$\begin{aligned}
P(X=x) &= \sum_{y=x}^{\infty} P(X=x, Y=y) \\
&= \sum_{y=x}^{\infty} 0 \\
&= 0.
\end{aligned}$$

Thus the marginal probability function of X is

$$P(X=x) = \begin{cases} 2^{-x} & \text{if } x \in \mathbb{N}, \\ 0 & \text{if } x \in \mathbb{Z}-\mathbb{N}. \end{cases}$$

When $y \in \mathbb{N}$, the marginal probability function of y is

$$\begin{aligned}
P(Y=y) &= \sum_{x=1}^{y} P(X=x, Y=y) \\
&= \sum_{x=1}^{y} c\times 2^{-y} \\
&= y2^{-1-y}.
\end{aligned}$$

When $x \in \mathbb{Z} - \mathbb{N}$, the marginal probability function of y is

$$\begin{aligned} P(Y=y) &= \sum_{x=1}^{y} P(X=x, Y=y) \\ &= \sum_{x=1}^{y} 0 \\ &= 0. \end{aligned}$$

Thus the marginal probability function of X is

$$P(Y=y) = \begin{cases} y2^{-1-y} & \text{if } y \in \mathbb{N}, \\ 0 & \text{if } y \in \mathbb{Z} - \mathbb{N}. \end{cases}$$

Since $P(X=x, Y=y) = P(X=x)P(Y=y)$ does not hold for every $(x,y) \in \mathbb{Z}^2$, X and Y are not independent.

$$\begin{aligned} P(X=Y) &= \sum_{k=1}^{\infty} P(X=Y=k) \\ &= \sum_{k=1}^{\infty} c2^{-k} \\ &= c \\ &= 1/2. \end{aligned}$$

3 Solution to Question 6:

Denote the marginal density of X_1 as $f_{X_1}(x_1)$. By definition, $f_{X_1}(x_1) = \int_{\mathbb{R}} f(x_1,x_2)dx_2$. When $x_1 < 0$, $f(x_1,x_2) = 0$, thus $f_{X_1}(x_1) = 0$. When $x_1 \geq 0$,

$$\begin{aligned} f_{X_1}(x_1) &= \int_{\mathbb{R}} f(x_1,x_2)dx_2 \\ &= \int_{-\infty}^{\infty} f(x_1,x_2)dx_2 \\ &= \int_0^{\infty} [(1+\alpha x_1)(1+\alpha x_2) - \alpha] exp\{-x_1 - x_2 - \alpha x_1 x_2\} dx_2 \\ &= e^{-x_1} \int_0^{\infty} [(1+\alpha x_1)(1+\alpha x_2) - \alpha] exp\{-(1+\alpha x_1)x_2\} dx_2 \\ &= e^{-x_1}(-1/(1+\alpha x_1)) \int_0^{\infty} [(1+\alpha x_1)(1+\alpha x_2) - \alpha] d exp\{-(1+\alpha x_1)x_2\} \\ &= e^{-x_1}(-1/(1+\alpha x_1))\{[(1+\alpha x_1)(1+\alpha x_2) - \alpha] exp\{-(1+\alpha x_1)x_2\}|_0^{\infty} \\ &- \int_0^{\infty} exp\{-(1+\alpha x_1)x_2\} d[(1+\alpha x_1)(1+\alpha x_2) - \alpha]\} \\ &= e^{-x_1}(-1/(1+\alpha x_1))\{-(1+\alpha x_1) + \alpha - \alpha \int_0^{\infty} exp\{-(1+\alpha x_1)x_2\} d(1+\alpha x_1)x_2\} \\ &= e^{-x_1}(-1/(1+\alpha x_1))\{-(1+\alpha x_1) + \alpha - \alpha \int_0^{\infty} exp\{-x_2\} dx_2\} \\ &= e^{-x_1}(-1/(1+\alpha x_1))\{-(1+\alpha x_1) + \alpha - \alpha \times 1\} \\ &= e^{-x_1} \end{aligned}$$

Thus the marginal density of X_1 is

$$f_{X_1}(x_1) = \begin{cases} e^{-x_1} & \text{if } x_1 \geq 0 \\ 0 & \text{if } x_1 < 0. \end{cases}$$

By symmetry, the marginal density of X_2 is

$$f_{X_2}(x_2) = \begin{cases} e^{-x_2} & \text{if } x_2 \geq 0 \\ 0 & \text{if } x_2 < 0. \end{cases}$$

Since $f(x, y) = f_{X_1}(x_1)f_{X_2}(x_2)$ does not hold every $(x, y) \in \mathbb{R}^2$, X_1 and X_2 are not independent.

4 Solution to Question 8:

Denote the joint probability mass function of (X, Y) as $f(x, y)$. The probability that the equation $z^2+2Xz+Y = 0$ has real roots (in z) is

$$\begin{aligned} P((2X)^2 - 4Y \geq 0) &= P(X^2 \geq Y) \\ &= \int_{x^2 \geq y} f(x, y)dxdy \\ &= \int_0^{\sqrt{\gamma}} \int_0^{x^2} f(x, y)dydx + \int_{\sqrt{\gamma}}^{\gamma} \int_0^{\gamma} f(x, y)dydx \\ &= \gamma^{-2}(\int_0^{\sqrt{\gamma}} \int_0^{x^2} dydx + \int_{\sqrt{\gamma}}^{\gamma} \int_0^{\gamma} dydx) \\ &= \gamma^{-2}((1/3)\gamma^{3/2} + \gamma(\gamma - \sqrt{\gamma})) \\ &= 1 - (2/3)\gamma^{-1/2} \\ &\to 1 \ \ as \ \ \gamma \to \infty. \end{aligned}$$

5 Solution to Question 10:

Suppose the lifetime of a light globe is $X \sim exp(1)$. Then $P(X \leq t) = 1 - e^{-t}$ and $P(X > t) = e^{-t}$.
For any $t > 0$,

$$\begin{aligned} P(T \leq t) &= P(\text{only 2 globe is burning}) \\ &+ P(\text{only 1 globes is burning}) \\ &+ P(\text{only 0 globe is burning}) \\ &= \binom{10}{2} P(X \leq t)^8 P(X > t)^2 + \binom{10}{1} P(X \leq t)^9 P(X > t)^1 + \binom{10}{0} P(X \leq t)^{10} \\ &= \binom{10}{2}(1 - e^{-t})^8(e^{-t})^2 + \binom{10}{1}(1 - e^{-t})^9(e^{-t})^1 + \binom{10}{0}(1 - e^{-t})^{10} \end{aligned}$$

Thus $P(T \leq 2) = \binom{10}{2}(1 - e^{-2})^8(e^{-2})^2 + \binom{10}{1}(1 - e^{-2})^9(e^{-2})^1 + \binom{10}{0}(1 - e^{-2})^{10} \approx 0.856$.

The density of T is just $\frac{d}{dt}P(T \leq t)$ for $t > 0$ and is 0 for $t \leq 0$, where

$$\frac{d}{dt}P(T \leq t) = 360(1 - e^{-t})^7 e^{-3t}.$$

6 Solution to Question 12:

In Problem 4,

$$\begin{aligned} P(Y \leq 1|X=1) &= P(Y \leq 1, X=1)/P(X=1) \\ &= P(Y=1, X=1)/P(X=1) \\ &= c2^{-1}/2^{-1} \\ &= c \\ &= 1/2. \end{aligned}$$

Next let's compute $P(X_2 \leq 1|X_1 = 1)$ in Problem 6. Denote $g(x_2|x_1)$ as the conditional density of X_2 given X_1. Then

$$g(x_2|x_1) = \begin{cases} f(x_1, x_2)/f_{X_1}(x_1) & \text{if } x_1 \geq 0 \\ 0 & \text{if } x_1 < 0. \end{cases}$$

That is,

$$g(x_2|x_1) = \begin{cases} [(1+\alpha x_1)(1+\alpha x_2) - \alpha]exp\{-x_2 - \alpha x_1 x_2\} & \text{if } x_1 \geq 0 \text{ and } x_2 \geq 0 \\ 0 & \text{orhterwise.} \end{cases}$$

When $x_1 = 1$,

$$g(x_2|1) = \begin{cases} [(1+\alpha)(1+\alpha x_2) - \alpha]exp\{-(1+\alpha)x_2\} & \text{if } x_2 \geq 0 \\ 0 & \text{orhterwise.} \end{cases}$$

Thus

$$\begin{aligned} P(X_2 \leq 1|X_1 = 1) &= \int_{-\infty}^{1} g(x_2|1)dx_2 \\ &= \int_0^1 [(1+\alpha)(1+\alpha x_2) - \alpha]exp\{-(1+\alpha)x_2\}dx_2 \\ &= 1 - (1+\alpha)exp\{-(1+\alpha)\}. \end{aligned}$$