

Math/Stat 425

Problem Set 4

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Note: For each problem, just one possible method for finding the solution is provided. Your method may be different.

Solution to Question 2:

We define Y = number of people who draw their own names

Let $X_i = 1$ if i th person draws their own name and 0 otherwise.

Then we have $Y = \sum_{i=1}^n X_i$

We will find the mean and variance of X_i and use it to find mean and variance of Y .

$$P(X_i = 1) = \frac{1}{n} \quad E(X_i) = \frac{1}{n}$$

Also

$$P(X_i X_j = 1) = P(X_i = 1)P(X_j = 1 | X_i = 1) = \frac{1}{n(n-1)} \quad E(X_i X_j) = \frac{1}{n(n-1)} \quad i \neq j.$$

Using this information we have:

$$\begin{aligned} E(Y) &= E\left(\sum_{i=1}^n X_i\right) \\ &= \sum_{i=1}^n E(X_i) \\ &= \sum_{i=1}^n \frac{1}{n} = 1 \end{aligned}$$

$$\begin{aligned}
\text{Var}(Y) &= E(Y^2) - E(Y)^2 \\
&= E\left(\sum_{i=1}^n X_i^2\right) - 1^2 \\
&= E(nX_i^2 + 2 \cdot \frac{n(n-1)}{2} X_i X_j) - 1 \quad i \neq j \\
&= nE(X_i^2) + n(n-1)E(X_i X_j) - 1 \\
&= 1 + 1 - 1 = 1
\end{aligned}$$

So the mean and variance of Y is 1.

Solution to Question 4:

$$Y = \min(X_1, \dots, X_n)$$

$$\begin{aligned}
F(y) = P(Y \leq y) &= 1 - P(Y > y) = 1 - [P(X > y)]^n \\
&= 1 - \left[\int_y^\infty \lambda e^{-\lambda x} dx\right]^n \\
&= 1 - e^{-n\lambda y} \quad y > 0
\end{aligned}$$

We can see this is a cumulative distribution function for an $\text{Exp}(n\lambda y)$ so the mean and variance of this distribution is $\frac{1}{n\lambda y}$ and $\frac{1}{(n\lambda y)^2}$

Solution to Question 6:

We have Profit X = Sales - Expenses.
The expenses are fixed daily we have that
Expenses = $50 \times 0.25 = 12.50$ dollars
Y = daily demand for doughnuts
Sold = $Y \cdot 1_{\{Y \leq 50\}} + 50 \cdot 1_{\{Y > 50\}}$

(where 1_A is indicator function equals 1 on the set A and 0 otherwise.)

$$E[\text{sold}] = \frac{1}{100} \sum_{i=1}^{100} i 1_{\{Y \leq 50\}} + \frac{1}{100} \sum_{i=1}^{100} 50 = 37.75 \quad \text{Expected(Sales)} = 0.5 \cdot E[\text{sold}] = 18.875 \text{ dollars}$$

Using this information we have
 $E[X] = E[\text{Profit}] = 6.375$ dollars

Solution to Question 8:

For our linear predictor $\tilde{Y} = b + aX$
Using our least squares estimate for a and b we get the following values.

$$\begin{aligned}
a &= \frac{\text{Cov}(X, Y)}{\text{Var}(X)} \\
&= \frac{\text{Cov}(Y+Z, Y)}{\text{Var}(Y+Z)} \\
&= \frac{\text{Cov}(Y, Y) + \text{Cov}(Z, Y)}{\text{Var}(Y) + \text{Var}(Z) + 2\text{Cov}(Y, Z)}
\end{aligned}$$

$$= \frac{\text{Var}(Y)}{\text{Var}(Y) + \text{Var}(Z)}$$

$$= \frac{5}{6}$$

$$b = \mu_Y - a\mu_X$$

$$= 10 - \frac{5}{6} \cdot 10 = \frac{10}{6}$$

So our estimate is $\tilde{Y} = \frac{5}{6}X + \frac{10}{6}$

If we had Y and $Z + 1$ are gamma distribution our estimate would remain unchanged.

Solution to Question 10:

From previous question we have that

$$f(z, w) = \frac{1}{2}e^{-w} \quad |z| \leq w$$

To find $f(z)$ We will find the cdf of Z

$$f(z) = \int_{|z|}^{\infty} f(z, w)dw$$

$$= \int_{|z|}^{\infty} \frac{1}{2}e^{-w}$$

$$= \frac{1}{2}e^{-|z|}$$

$$f(w|z) = \frac{f(z, w)}{f(z)} = e^{|z|-w}$$

$$E[W|Z = z] = \int_{|z|}^{\infty} wf(w|z)dw$$

$$= \int_{|z|}^{\infty} we^{|z|-w}dw$$

$$= |z| + 1$$

$$E[W^2|Z = z] = \int_0^{\infty} w^2 f(w|z)dw$$

$$= \int_0^{\infty} w^2 e^{|z|-w}dw$$

$$= |z|^2 + 2|z| + 2$$

Using the above information we have the conditional mean of W given $Z=z$ is $|z| + 1$
the conditional variance of W given $Z=z$ is 1

Solution to Question 12:

$Z = Y - X$ where X and Y are both $\text{Exp}(\lambda = 1)$.

The moment generating function for X and Y is given by $\varphi_X(t) = \frac{1}{1-t}$.
Using properties of Moment generating functions we have the the moment generating funtion for Z is

$$\begin{aligned}\varphi_Z(t) &= \varphi_{X-Y}(t) \\ &= \varphi_X(t) \varphi_Y(-t) \\ &= \frac{1}{1-t} \cdot \frac{1}{1+t} = \frac{1}{1-t^2}\end{aligned}$$

We can find the first four moments my taking the derivative of the moment generating funtion to get the following.

$$\begin{aligned}\varphi_Z(t) &= \frac{1}{1-t^2} \\ \varphi_Z'(t) &= \frac{2t}{(1-t^2)^2} \\ \varphi_Z^{(2)}(t) &= \frac{8t^2}{(1-t^2)^3} + \frac{2}{(1-t^2)} \\ \varphi_Z^{(3)}(t) &= \frac{48t^3}{(1-t^2)^4} + \frac{24t}{(1-t^2)} \\ \varphi_Z^{(4)}(t) &= \frac{384t^4}{(1-t^2)^5} + \frac{288t^2}{(1-t^2)} + \frac{24}{(1-t^2)}\end{aligned}$$

$$\begin{aligned}\text{Var}(Z^2) &= E(Z^4) - E(Z^2)^2 \\ &= \varphi_Z^{(4)}(0) - (\varphi_Z^{(2)}(0))^2 \\ &= 24 - 2^2 = 20\end{aligned}$$