The Martingale Central Limit Theorem

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Preliminaries-Martingale Differences. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \mathcal{A}$ be a (finite or infinite) sequence of increasing sigma-algebras. Then a (finite or infinite) collection $X_1, X_2, \cdots$ of random variables are said to be *martingale differences with respect to* $\mathcal{A}_0, \mathcal{A}_1, \cdots$ if

- X_k is $\mathcal{A}_k$-measurable;
- $E|X_k| < \infty$;
- $E(X_k|\mathcal{A}_{k-1}) = 0$ $w.p.1$

for all k. If martingale differences $X_1, X_1, \cdots$ are square integrable, then they are are orthogonal. For, if $j < k$, then $E(X_j X_k) = E[E(X_j X_k|\mathcal{A}_j)] = E[X_j \times E(X_k|\mathcal{A}_j)] = 0$.

Triangular Arrays. Consider a triangular array of square integrable martingale differences

$$\{X_{nk}, \mathcal{A}_{nk},\ k = 1, \cdots, k_n\}.$$

Thus, $\mathcal{A}_{n0} \subseteq \mathcal{A}_{n1} \subseteq \cdots \subseteq \mathcal{A}_{nk_n}$ are subsigma algebras, and $X_{n1}, \cdots, X_{nk_n}$ are square integrable random variables for which X_{nk} is $\mathcal{A}_{nk}$-measurable and $E(X_{nk}|\mathcal{A}_{n,k-1}) = 0$ $w.p.1$ for all $k = 1, \cdots, n$. Conditions are sought under which

$$S_n^* = X_{n1} + \cdots + X_{nk_n}$$

are asymptotically normal. For the remainder of today $\{X_{nk}, \mathcal{A}_{nk},\ k = 1, \cdots, k_n\}$ denote square integrable martingale differences.

For examples of such arrays, let $S_0, S_1, \cdots$ be a square martingale with respect to a filtration $\mathcal{A}_0, \mathcal{A}_1, \cdots$, and let $X_k = S_k - S_{k-1}$. Then the conditions are satisifed by

$$X_{nk} = \frac{1}{\sqrt{n}} X_k \qquad \text{and} \qquad \mathcal{A}_{nk} = \mathcal{A}_k$$

for $k = 1, \cdots, n$. For a more specific example, let $Z_0, Z_1, Z_2, \cdots$ be i.i.d. with mean 0 and finite positive variance; let $\mathcal{A}_k = \sigma\{Z_0, \cdots, Z_k\}$; and let $X_k = Z_{k-1} Z_k,\ k = 1, 2, \cdots$.

Notation. Now let

$$\sigma_{nk}^2 = E(X_{nk}^2|\mathcal{A}_{n,k-1}),$$
$$V_{nk} = \sigma_{n1}^2 + \cdots + \sigma_{nk}^2,$$

and

$$V_n = V_{nk_n} = \sigma_{n1}^2 + \cdots + \sigma_{nk_n}^2$$

for $k = 1, \cdots, k_n$, and observe that σ_{nk}^2 and V_{nk} are both $\mathcal{A}_{n,k-1}$-measurable.

Two conditions are required for the main result. To state the first, let

$$L_n(\epsilon) = \sum_{k=1}^{k_n} E\big[X_{nk}^2 \mathbf{1}_{\{|X_{nk}| \ge \epsilon\}} | \mathcal{A}_{n,k-1}\big]$$

and observe that

$$E[L_n(\epsilon)] = \sum_{k=1}^{k_n} E\big[X_{nk}^2 \mathbf{1}_{\{|X_{nk}|\geq\epsilon\}}\big].$$

Lindeberg-Feller Condition. For every $\epsilon > 0$,

$$\lim_{n\to\infty} L_n(\epsilon) = 0$$

in probability. Observe that this is implied by the stronger condition

$$\lim_{n\to\infty} E[L_n(\epsilon)] = 0$$

and that the two are equivalent if V_n are uniformly bounded.

The Stability Condition. There is a positive constant η for which

$$\lim_{n\to\infty} V_n = \eta.$$

Lemma 1. *If the Lindeberg-Feller Condition is satisfied, then*

$$\lim_{n\to\infty} \max_{k\leq k_n} \sigma_{nk}^2 = 0$$

in probability.

Proof. For any $0 < \epsilon < 1/2$,

$$\max_{k\leq k_n} \sigma_{nk}^2 \leq \epsilon^2 + L_n(\epsilon) \leq \frac{1}{2}\epsilon + L_n(\epsilon)$$

So,

$$P\{\max_{k\leq k_n} \sigma_{nk}^2 \geq \epsilon\} \leq P\{L_n(\epsilon) \geq \frac{1}{2}\epsilon\},$$

which approaches zero as $n \to \infty$. ◊

Lemma 2. *Suppose that the Stability Condition is satisfied and let*

$$X'_{nk} = X_{nk}\mathbf{1}_{\{V_{nk}\leq 2\eta\}}, \;\; k = 1, \cdots, k_n.$$

Then $\{X'_{nk}, \mathcal{A}_{nk}, \;\; k = 1, \cdots, k_n\}$ *are square integrable martingale differences for which*

$$V'_n \leq 2\eta,$$
$$\lim_{n\to\infty} P\{X'_{nk} = X_{nk} \text{ for all } k = 1, \cdots, k_n\} = 1,$$
$$\lim_{n\to\infty} P\{V'_n = V_n\} = 1.$$

Further, if $\{X_{nk}, \mathcal{A}_{nk},\ k = 1, \cdots, k_n\}$ satisfy the Lindeberg Feller Condition, then so do $\{X'_{nk}, \mathcal{A}_{nk},\ k = 1, \cdots, k_n\}$.

Proof. The X'_{nk}s are martingale differences, because

$$E\big[X'_{nk}|\mathcal{A}_{nk}\big] = \mathbf{1}_{\{V_{nk} \le 2\eta\}} E\big[X_{nk}|\mathcal{A}_{nk}\big] = 0$$

for all $k = 1, \cdots, k_n$. Since $|X'_{nk}| \le |X_{nk}|$, is clear that X'_{nk} are square integrable, $L'_n(\epsilon_\le L_n(\epsilon)$, and

$$\sigma'^2_{nk} = \mathbf{1}_{\{V_{nk} \le 2\eta\}} \sigma^2_{nk},$$

as above. If J is the largest $k \le k_n$ for which $V_{nk} \le 2\eta$, then $V'_{nk} = V_{n,k\wedge J} \le 2\eta$ for all $k \le k_n$. Finally, if $V_n \le 2\eta$, then $X'_{nk} = X_{nk}$ for $k \le n$ and $V'_n = V_n$; and $\lim_{n\to\infty} P[V_n \le 2\eta] = 1$.

Recall that

$$|\rho_k(t)| = |e^{\imath t} - \sum_{j=0}^{k-1} \frac{1}{j!}(\imath t)^j| \le \frac{1}{k!}|t|^k$$

for all $t \in R$ and $k = 1, 2, \cdots$. Similarly, $|e^{-t} - 1 + t| \le t^2/2$ for $t \ge 0$.

A Martingale Central Limit Theorem. *If the Stability and Lindeberg Feller Conditions are satisfied, then*

$$S^*_n \Rightarrow Z \sim \text{Normal}(0, \eta).$$

Proof. There is no loss of generalty in supposing that $\eta = 1$ and that $V_n \le 2$ *w.p.*1. Under these conditions, it will be shown that

$$\lim_{n\to\infty} E[e^{\imath t S^*_n}] = e^{-\frac{1}{2}t^2}$$

for all t. Fix t and write

$$E[e^{\imath t S^*_n}] - e^{-\frac{1}{2}t^2} = e^{-\frac{1}{2}t^2} E\big[R_{n,1}(t) + R_{n,2}(t)\big],$$

where

$$R_{n,1}(t) = e^{\imath t S^*_n}\big[e^{\frac{1}{2}t^2} - e^{\frac{1}{2}V_n t^2}\big]$$

and

$$R_{n,2}(t) = e^{\imath t S^*_n + \frac{1}{2}V_n t^2} - 1\big].$$

Clearly,

$$E|R_{n,1}| \le E|e^{\frac{1}{2}t^2} - e^{\frac{1}{2}V_n t^2}| \to 0,$$

since V_n are bounded and, therefore, uniformly integrable. To estimate the second term, write

$$\begin{aligned} R_{n,2}(t) &= \sum_{k=1}^{k_n} \big[e^{\imath t S_{nk} + \frac{1}{2}V_{nk}t^2} - e^{\imath t S_{n,k-1} + \frac{1}{2}V_{n,k-1}t^2}\big] \\ &= \sum_{k=1}^{k_n} e^{\imath t S_{n,k-1} + \frac{1}{2}V_{nk}t^2}\big[e^{\imath t X_{nk}} - e^{-\frac{1}{2}\sigma^2_{nk}t^2}\big] \end{aligned}$$

Here

$$\begin{aligned}
&\big|E\big\{e^{\imath t S_{n,k-1}+\frac{1}{2}V_{n,k}t^2}\big[e^{\imath t X_{nk}}-e^{-\frac{1}{2}\sigma_{nk}^2 t^2}-1\big]\big\}\big| \\
&\qquad = \big|E\big\{e^{\imath t S_{n,k-1}+\frac{1}{2}V_{nk}t^2}E\big[e^{\imath t X_{nk}}-e^{-\frac{1}{2}\sigma_{nk}^2 t^2}|\mathcal{A}_{n,k-1}\big]\big\}\big| \\
&\qquad \le e^{t^2}E\big|E\big[e^{\imath t X_{nk}}-e^{-\frac{1}{2}\sigma_{nk}^2 t^2}|\mathcal{A}_{n,k-1}\big]\big|
\end{aligned}$$

for all $k = 1, \cdots, k_n$, since $V_n \le 2$ $w.p.1$. Next,

$$e^{\imath t X_{nk}} = 1 + \imath t X_{nk} - \frac{1}{2}t^2 X_{nk}^2 + \rho_3(tX_{nk})$$

and

$$e^{-\frac{1}{2}\sigma_{nk}^2 t^2} = 1 - \frac{1}{2}\sigma_{nk}^2 t^2 + \tilde{\rho}_2(-\frac{1}{2}\sigma_{nk}^2 t^2),$$

where

$$|\rho_3(tX_{nk})| \le \min[t^2 X_{nk}^2,\ \frac{1}{6}|t^3 X_{nk}^3|]$$

and

$$|\tilde{\rho}_2(-\frac{1}{2}\sigma_{nk}^2 t^2)| \le \frac{1}{4}\sigma_{nk}^4 t^4.$$

So, since $E(X_{nk}|\mathcal{A}_{n,k-1}) = 0$ and $E(X_{nk}^2|\mathcal{A}_{n,k-1}) = \sigma_{nk}^2$,

$$E\big[e^{\imath t X_{nk}} - e^{-\frac{1}{2}\sigma_{nk}^2}|\mathcal{A}_{n,k-1}\big] = E\big[\rho_3(tX_{nk}) - \tilde{\rho}_2(-\frac{1}{2}\sigma_{nk}^2)|\mathcal{A}_{n,k-1}\big]$$

and, therefore,

$$|E[R_{n,2}(t)]| \le e^{t^2}\sum_{k=1}^{k_n} E\big[|\rho_3(tX_{nk})| + |\tilde{\rho}_2(-\frac{1}{2}\sigma_{nk}^2)|\big].$$

Here,

$$\sum_{k=1}^{k_n} E\big[|\tilde{\rho}_2(-\frac{1}{2}\sigma_{nk}^2)|\big] \le \frac{1}{4}t^4 E\big[\max_{k\le k_n}\sigma_{nk}^2 \times V_n\big] \to 0,$$

as $n \to \infty$, since $\max_{k\le k_n}\sigma_{nk}^2 \times V_n \to 0 \times \eta = 0$ in probability and $\max_{k\le k_n}\sigma_{nk}^2 V_n \le V_n^2 \le 4$ $w.p.1$. Next, for any $\epsilon > 0$,

$$\begin{aligned}
E|\rho_3(tX_{nk})| &\le \frac{1}{6}|t|^3\int_{\{|X_{nk}|\le\epsilon\}}|X_{nk}|^3 dP + t^2\int_{\{|X_{nk}|>\epsilon\}}|X_{nk}|^2 dP \\
&\le \frac{1}{6}\epsilon|t|^3\int_{\{|X_{nk}|^2\le\epsilon\}}|X_{nk}|dP + t^2\int_{\{|X_{nk}|>\epsilon\}}|X_{nk}|^2 dP
\end{aligned}$$

so that

$$\sum_{k=1}^{k_n} E|\rho_3(tX_{nk})| \le\le \frac{1}{6}\epsilon|t|^3 E[V_n] + t^2 E[L_n(\epsilon)]$$

and, therefore,

$$\limsup_{n\to\infty}\sum_{k=1}^{k_n} E|\rho_3(tX_{nk})| \le \frac{1}{3}\epsilon|t|^3.$$

That

$$\lim_{n\to\infty}|E[R_{n,2}]| = 0$$

follows easily, since $\epsilon > 0$ was arbitrary, completing the proof of the theorem. $\diamondsuit$

Example: Autoregressive Processes. Let $\cdots Z_{-1}, Z_0, Z_1, Z_2, \cdots$ be i.i.d. random variables with mean 0 and a finite positive variance σ^2; let $-1 < \theta < 1$; and let

$$Y_k = \sum_{j=0}^{\infty}\theta^j Z_{k-j}$$

for $k \in \mathbb{Z}$. The sum converges *w.p.*1 for each k, by the Two Series Theorem, and

$$E(Y_k^2) = \sum_{j=0}^{\infty}\theta^{2j}E(Z_{k-j}^2) = \frac{\sigma^2}{1-\theta^2}.$$

It is easily seen that

$$Y_k = \theta Y_{k-1} + Z_k$$

for $k = 1, 2, \cdots$, and $\cdots Y_{-1}, Y_0, Y_1, Y_2, \cdots$. For this reason $Y_0, Y_1, \cdots$ is called an *auto regressive process.* If $Y_0, \cdots, Y_n$ are observed, then the least squares estimator of θ is

$$\hat\theta_n = \frac{\sum_{k=1}^n Y_{k-1}Y_k}{\sum_{k=1}^n Y_{k-1}^2}$$

Observe that

$$\hat\theta_n - \theta = \frac{\sum_{k=1}^n Y_{k-1}Z_k}{\sum_{k=1}^n Y_{k-1}^2}.$$

It will be shown that $\sqrt{n}(\hat\theta_n - \theta)$ is asymptotically normal, as a consequence of the last theorem. In verifying the conditions, I will anticipate the *Ergodic Theorem*: if $E|g(Y_0)| < \infty$, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n g(Y_k) = E[g(Y_0)].$$

It follows that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n Y_k^2 = \frac{\sigma^2}{1-\theta^2}\ \ w.p.1.$$

Let

$$X_{nk} = \frac{1}{\sqrt{n}}Y_{k-1}Z_k$$

for $k = 1, \cdots, n$. Then $X_{n1}, \cdots, X_{nn}$ are martingale differences with respect to $\mathcal{A}_k = \sigma\{Y_0, Z_1, \cdots, Z_k\}$, since

$$E(X_{nk}|\mathcal{A}_k) = \frac{1}{\sqrt{n}} Y_{k-1} E(Z_k) = 0$$

for $k = 1, \cdots, n$. Moreover,

$$\sigma_{nk}^2 = \frac{1}{n} Y_{k-1}^2 E(Z_k^2) = \frac{\sigma^2}{n} Y_{k-1}^2$$

and

$$V_n = \frac{\sigma^2}{n} \sum_{k=1}^{n} Y_{k-1}^2 \to \frac{\sigma^4}{1-\theta^2} \ w.p.1.$$

So, the Stability Condition is satisfied. For the Lindeberg Feller Condition

$$\sum_{k=1}^{n} E[X_{nk}^2 \mathbf{1}_{\{|X_{nk}| \geq \epsilon\}}] = E[Y_0^2 Z_1^2 \mathbf{1}_{\{|Y_0 Z_1| \geq \epsilon \sqrt{n}\}}] \to 0,$$

by the Dominated Convergence Theorem. So,

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_{k-1} Z_k = X_{n1} + \cdots + X_{nn} \Rightarrow \text{Normal}[0, \frac{\sigma^4}{1-\theta^2}]$$

and

$$\sqrt{n}(\hat{\theta}_n - \theta) = \frac{\sum_{k=1}^{n} Y_{k-1} Z_k / \sqrt{n}}{\sum_{k=1}^{n} Y_{k-1}^2 / n} \Rightarrow \text{Normal}(0, 1-\theta^2).$$