Isotonic Estimation: The Asymptotic Distribution

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Isotonic Regression. Consider an isotonic regression model

$$y_k = \phi(\frac{k}{n}) + \epsilon_k,\ k = 1, \cdots, n,$$

where ϕ is non-dcreasing, and $\epsilon_1, \cdots, \epsilon_n$ are i.i.d. errors with mean 0, finite variance σ^2, and a finite moments generating function near 0. Let $G_n^\#$ denote the normalized cummulative sum diagram; thus, $G_n^\#$ is a continuous piecewise linear function for which

$$G_n^\#(\frac{k}{n}) = \frac{1}{n}\sum_{j=1}^{k} y_j$$

for $j = 1, \cdots, n$. Also, let

$$\Phi(t) = \int_0^t \phi(s) ds$$

and let $\Phi^\#$ be a continuous piecewise linear function for which

$$\Phi_n^\#(\frac{k}{n}) = \frac{1}{n}\sum_{j=1}^{k} \phi(\frac{j}{n}).$$

Then, under modest conditions (in particular, if ϕ has a bounded derivative),

$$\sup_t |\Phi_n^\#(t) - \Phi(t)| = O(\frac{1}{n});$$

and then

$$G_n^\#(t) = \Phi(t) + \frac{\sigma}{\sqrt{n}} \mathbb{B}_n(t) + R_n(t),$$

where $\mathbb{B}_n$ is a Brownian motion and

$$\sup_t |R_n(t)| = O[\frac{\log(n)}{n}]\ w.p.1.$$

Now, fix a $0 < t < 1$ and suppose that ϕ has a positive continuous derivative ϕ' on some neighborhood of t. Then

$$G^{\#}(t+n^{-\frac{1}{3}}s) - G^{\#}(t) - \phi(t)n^{-\frac{1}{3}}s = [\Phi(t+n^{-\frac{1}{3}}s) - \Phi(t) - \phi(t)n^{-\frac{1}{3}}s] + \frac{\sigma}{\sqrt{n}}[\mathbb{B}_n(t+n^{-\frac{1}{3}}s) - \mathbb{B}_n(t)] + [R_n(t+n^{-\frac{1}{3}}s) - R_n(t)],$$

Obserse that $W_n(s) = n^{\frac{1}{6}}[[\mathbb{B}_n(t+n^{\frac{1}{3}}s) - \mathbb{B}_n(t)]$ is a two-sided Brownian motion and let

$$Z_n(s) = n^{\frac{2}{3}}[G_n^{\#}(t+n^{-\frac{1}{3}}s) - G_n^{\#}(t) - \phi(t)n^{-\frac{1}{3}}s]$$

and

$$Z_n^o = \sigma W_n(s) + \frac{1}{2}\phi'(t)s^2,$$

and observe that the distribution of Z_n^o does not depend on n. Then

$$Z_n(s) = Z_n^o(s) + \gamma_n(s) + R_n^t(s)$$

where

$$R_n^t(s) = n^{\frac{2}{3}}[R_n(t+n^{-\frac{1}{3}}s) - R_n(t)] = O\left[\frac{\log(n)}{n^{1/3}}\right] \ w.p.1$$

and

$$\gamma_n(s) = n^{\frac{2}{3}}[\Phi(t+n^{-\frac{1}{3}}s) - \Phi(t) - \phi(t)n^{-\frac{1}{3}}s - \frac{1}{2}\phi'(t)n^{-\frac{2}{3}}s^2].$$

Here $\gamma_n(s) \to 0$ as $n \to \infty$ uniform on $|s| \le c$ for any $0 < c < \infty$ and, therefore, for some sequence $c = c_n \to \infty$. So

$$\sup_{|s|\le c_n} |Z_n(s) - Z_n^o(s)| \to^p 0.$$

Next, let $\tilde{G}_n$ be the greatest convex minorant of $G_n^{\#}$, and recall that the least square estimator of ϕ is $\phi_n(t) = \tilde{G}'_{n,\ell}(t)$. Recall too that if f is bounded and h is linear, then $(f+h)\tilde{} = \tilde{f} + h$. So,

$$\tilde{G}_n(t+n^{-\frac{1}{3}}s) - G_n^{\#}(t) - \phi(t)n^{-\frac{1}{3}}s = [\tilde{G}_n^{\#}(t+n^{-\frac{1}{3}}s) - G_n^{\#}(t) - \phi(t)n^{-\frac{1}{3}}s].$$

Let $\tilde{Z}_n$ and $\tilde{Z}_n^o$ denote the greatest convex minorants of the restrictions of Z_n and Z_n^o to $|s| \le c_n$. Then with probability approaching one

$$n^{\frac{2}{3}}[\tilde{G}_n(t+n^{-\frac{1}{3}}s) - G_n^{\#}(t) - \phi(t)n^{-\frac{1}{3}}s] = \tilde{Z}_n(s)$$

and

$$\sup_{|s|\le c_n} |\tilde{Z}_n(s) - \tilde{Z}_n^o(s)| \le \sup_{|s|\le c_n} |Z_n(s) - Z_n^o(s)| \to^p 0$$

Finally, let $W(s), -\infty < s < \infty$, be a standard two-sided Brownian motion, $Z_{a,b}(s) = aW(s) + bs^2$, $Z(s) = Z_{1,1}$, and let $\tilde{Z}_{a,b}$ be the greatest convex minorant (on $I\!R$) of $Z_{a,b}$. Then, using the fact that the distribution of Z_n^o is the same as the distribution of $Z_{\sigma,\frac{1}{2}\phi'(t)}$ the restrictions of $\tilde{Z}_n$ and $\tilde{Z}_n^o$ to any finite interval converge to those of (the restriction) of $\tilde{Z}_{\sigma,\frac{1}{2}\phi'(t)}$. In particular,

$$\tilde{Z}_n|_{[-1.1]} \Rightarrow \tilde{Z}_{\sigma,\frac{1}{2}\phi'(t)}|_{[-1,1]}.$$

As a corollary

$$n^{\frac{1}{3}}[\hat{\phi}_n(t) - \phi(t)] = n^{\frac{1}{3}}[\tilde{G}'_{n,\ell}(t) - \phi(t)] = \tilde{Z}'_{n,\ell}(0)$$

with probability approaching one. So,

$$n^{\frac{1}{3}}[\hat{\phi}_n(t) - \phi(t)] \Rightarrow \tilde{Z}'_{\sigma,\frac{1}{2}\phi'(t)}(0),$$

by the continuous mapping principle.

On the Limiting Distribution. Let $Z_*(t) = Z_{a,b}(ct)$, $-\infty < t < \infty$. Then, by rescaling, $Z_*(s) = aW(cs) + b(cs)^2$ has the same distribution as $a\sqrt{c}W(s) + bc^2s^2$. So, letting $c = (a/b)^{2/3}$, $Z_{a,b} =^{\mathcal{D}} a^{\frac{4}{3}}b^{-\frac{1}{3}}Z$. Letting $a = \sigma$ and $b = \phi'(t)/2$, it follows that

$$\tilde{Z}_n|_{[-1.1]} \Rightarrow a^{\frac{4}{3}}b^{-\frac{1}{3}}\tilde{Z}|_{[-1,1]},$$

$$\tilde{Z}'_n(0) \Rightarrow \frac{1}{c}a^{\frac{4}{3}}b^{-\frac{1}{3}}\tilde{Z}'(0) = (a^2b)^{\frac{1}{3}}\tilde{Z}'(0),$$

and

$$n^{\frac{1}{3}}[\hat{\phi}_n(t) - \phi(t)] \Rightarrow \kappa\tilde{Z}'(0),$$

where

$$\kappa = \left[\frac{1}{2}\sigma^2\phi'(t)\right]^{\frac{1}{3}}.$$

The distribution of $\tilde{Z}'(0)$ first appeared in [1] in a different context. It is a tabled in [2] which can be consulted for a (slighlty biased collection of) further references. The distribution has shorter tails than the standard normal.

References

[1] Chernoff, Herman (1964). Estimation of the mode. *Ann. Math. Statist.*, **16**, 31-41.

[2] Groeneboom, Piet and Jon Wellner (2001). Computing Chernoff's Distribution. *JCGS*, **10**, 388-400.