

Assignment 1: Review of linear algebra

Due on Wednesday, Sep 17, 2008

1. (Vandermonde determinants) Show that

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \cdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

2. Calculate *by hand* the determinant and the inverse (if it exists) of the matrix:

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 0.6 & -0.4 \\ 0 & 0 & 0.2 & 0.2 \end{pmatrix}$$

3. (Schott, p. 128, Problem 3.1) Consider the 3×3 matrix:

$$A = \begin{pmatrix} 9 & -3 & -4 \\ 12 & -4 & -6 \\ 8 & -3 & -3 \end{pmatrix}$$

- (a) Find the eigen values of A .
(b) Find normalized vectors corresponding to each eigen value.
(c) Find $\text{trace}(A^{10}) = ?$
4. Let $A = (\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_k)$ and $B' = (\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_k)$, where $\vec{a}_i \in \mathbb{R}^m$ and $\vec{b}_i \in \mathbb{R}^n$ are *column vectors* of dimensions m and n , respectively. Therefore, A and B are matrices of dimensions $m \times k$ and $k \times n$, respectively. Prove rigorously that

$$AB = \vec{a}_1 \vec{b}_1' + \vec{a}_2 \vec{b}_2' + \cdots + \vec{a}_k \vec{b}_k'.$$

5. Let $A = (a_{ij})_{m \times m}$ be a square matrix. Show that $\det(A) \neq 0$ if and only if the column vectors of A are linearly independent.
6. Let $A = (a_{ij})_{m \times n}$. Show that:
(a) $A'A$ is invertible if and only if $n \leq m$ and $\text{rank}(A) = n$.
(b) $\text{rank}(A'A) = \text{rank}(A)$.
7. Let $A = (a_{ij})_{m \times k}$ and $B = (b_{ij})_{k \times n}$ be real matrices. If both A and B have full column ranks, then show that AB has full column rank.
8. (Schott, p. 130, Problem 3.8) If A and B are two $m \times m$ matrices and at least one of them is non-singular, then show that the eigen values of AB and BA are the same.

9. (a) Let $X = (x_{ij})_{m \times n}$ be a real matrix. Show that the matrix $A := X^t X$ is positive semi-definite;
- (b) Conversely, if A is a $n \times n$ real symmetric and positive semi-definite matrix, then show that $A = X^t X$, for some matrix X . Show moreover, that X can be chosen to be symmetric.
- (c) (Schott, p. 134, Problem 3.47) Show that if $A = (a_{ij})_{n \times n}$ is a symmetric, positive semi-definite matrix such that $a_{ii} = 0$, then $a_{ij} = a_{ji} = 0$ for all $1 \leq j \leq n$.
10. (Schott, p. 135, Problem 3.49) Suppose that A and B are two real symmetric $n \times n$ matrices with eigen values λ_j , $1 \leq j \leq n$ and μ_j , $1 \leq j \leq n$. Suppose that the matrices A and B can be diagonalized simultaneously by an orthogonal transformation. That is, there exist orthonormal $\vec{x}_j \in \mathbb{R}^n$, $1 \leq j \leq n$, which are eigen vectors of *both* A and B corresponding to the values λ_j and μ_j , $1 \leq j \leq n$, respectively.
- (a) Find the eigen values and the eigen vectors of $A + B$ and AB .
- (b) Show that $AB = BA$.
11. Let $A = (a_{ij})_{n \times n}$ be a real symmetric matrix with eigen values $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.
- (a) Show that

$$\lambda_1 = \sup_{x \in \mathbb{R}^n, x \neq \vec{0}} \frac{x^t A x}{x^t x}.$$

(b) Let B be a real symmetric matrix with eigen values $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$. If B is *non-singular* and $AB = BA$, then determine (in terms of the λ_i 's and the μ_i 's) the supremum

$$\sup_{x \in \mathbb{R}^n, x \neq \vec{0}} \frac{x^t A x}{x^t B x} = ?$$

12. Let $B = (b_{ij})_{n \times n}$ be a real, non-singular matrix and let $\|\cdot\|$ be an arbitrary matrix norm in the space of all $n \times n$ real matrices $M_n(\mathbb{R})$.
- a. Show that there exists $\epsilon > 0$, such that the matrix $B + A$ is invertible, for any $A \in M_n(\mathbb{R})$ with $\|A\| < \epsilon$.
- b. Given an arbitrary matrix $A \in M_n(\mathbb{R})$, show that $C_\delta := B + \delta A$ is invertible for some $\delta > 0$,
13. Let $Y \sim \mathcal{N}(\vec{0}, \Sigma)$, $\Sigma = (\sigma_{ij})_{n \times n}$ be a multivariate normal random vector in \mathbb{R}^n with covariance matrix Σ and zero mean. Show that there exist n orthonormal *constant* vectors $\vec{x}_1, \dots, \vec{x}_n \in \mathbb{R}^n$, such that

$$Y \stackrel{d}{=} \sqrt{\lambda_1} Z_1 \vec{x}_1 + \dots + \sqrt{\lambda_n} Z_n \vec{x}_n,$$

where λ_j , $j = 1, \dots, n$ are the eigen values of Σ and where Z_j , $j = 1, \dots, n$ are independent standard Normal random variables.

14. Let Z_j , $j = 1, \dots, n$ be independent standard Normal random variables and let $\vec{x}_1, \dots, \vec{x}_k \in \mathbb{R}^n$, be orthonormal *constant* vectors in \mathbb{R}^n , $1 \leq k < n$. Consider the random vector

$$X := Z - \vec{x}_1(\vec{x}_1^t Z) - \dots - \vec{x}_k(\vec{x}_k^t Z),$$

where $Z = (Z_1 \dots Z_n)^t$. Determine the probability distribution of the random variable $\xi := X^t X$.