

Assignment 1: Review of linear algebra

Due on Wed, Sep 30, 2009

1. Calculate *by hand* the determinant and the inverse (if it exists) of the matrix:

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & 0.2 & 0.2 \\ 0 & 0 & 0.6 & -0.4 \end{pmatrix}$$

2. (Schott, p. 128, Problem 3.1) Consider the 3×3 matrix:

$$A = \begin{pmatrix} 9 & -3 & -4 \\ 12 & -4 & -6 \\ 8 & -3 & -3 \end{pmatrix}$$

- (a) Find the eigen values of A .
(b) Find normalized vectors corresponding to each eigen value.
(c) Find $\text{trace}(A^{10}) = ?$

3. Let $A = (\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_k)$ and $B' = (\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_k)$, where $\vec{a}_i \in \mathbb{R}^m$ and $\vec{b}_i \in \mathbb{R}^n$ are *column vectors* of dimensions m and n , respectively. Therefore, A and B are matrices of dimensions $m \times k$ and $k \times n$, respectively. Prove rigorously that

$$AB = \vec{a}_1 \vec{b}_1' + \vec{a}_2 \vec{b}_2' + \cdots + \vec{a}_k \vec{b}_k'$$

4. (*Block-wise operations*) Let $A = (A_{ij})_{m \times p}$ and $B = (B_{ij})_{p \times n}$, where $A_{ij} = (a_{k\ell}(i, j))_{m_i \times p_j}$ and $B_{ij} = (b_{k\ell}(i, j))_{p_i \times n_j}$ are $m_i \times p_j$ and $p_i \times n_j$ matrices, respectively.

That is, the matrices A and B are divided into $m \times p$ and $p \times n$ blocks. Show that

$$AB = (C_{ij})_{m \times n},$$

where $C_{ij} = \sum_{k=1}^p A_{ik} B_{kj}$, is a $m_i \times n_i$ matrix.

5. (*Vandermonde determinants*) Show that

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \cdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

6. A square matrix $A = (a_{ij})_{n \times n}$ is said to be strictly diagonal-dominant if

$$\sum_{j \neq i} |a_{ij}| < |a_{ii}|, \quad \text{for all } 1 \leq i \leq n.$$

Prove that every such matrix is invertible.

7. Let $A = (a_{ij})_{m \times m}$ be a square matrix. Show that $\det(A) \neq 0$ if and only if the column vectors of A are linearly independent.
8. Let $A = (a_{ij})_{m \times n}$. Show that:
- $A'A$ is invertible if and only if $n \leq m$ and $\text{rank}(A) = n$.
 - $\text{rank}(A'A) = \text{rank}(A)$.
9. Let $A = (a_{ij})_{m \times k}$ and $B = (b_{ij})_{k \times n}$ be real matrices. If both A and B have full column ranks, then show that AB has full column rank.
10. (a) Let $X = (x_{ij})_{m \times n}$ be a real matrix. Show that the matrix $A := X^t X$ is positive semi-definite;
- (b) Conversely, if A is a $n \times n$ real symmetric and positive semi-definite matrix, then show that $A = X^t X$, for some matrix X . Show moreover, that X can be chosen to be symmetric.
- (c) (Schott, p. 134, Problem 3.47) Show that if $A = (a_{ij})_{n \times n}$ is a symmetric, positive semi-definite matrix such that $a_{ii} = 0$, then $a_{ij} = a_{ji} = 0$ for all $1 \leq j \leq n$.
11. (Schott, p. 135, Problem 3.49) Suppose that A and B are two real symmetric $n \times n$ matrices with eigen values λ_j , $1 \leq j \leq n$ and μ_j , $1 \leq j \leq n$. Suppose that the matrices A and B can be diagonalized simultaneously by an orthogonal transformation. That is, there exist orthonormal $\vec{x}_j \in \mathbb{R}^n$, $1 \leq j \leq n$, which are eigen vectors of *both* A and B corresponding to the values λ_j and μ_j , $1 \leq j \leq n$, respectively.
- Find the eigen values and the eigen vectors of $A + B$ and AB .
 - Show that $AB = BA$.
12. Let $A = (a_{ij})_{n \times n}$ be a real symmetric matrix with eigen values $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.
- (a) Show that

$$\lambda_1 = \sup_{x \in \mathbb{R}^n, x \neq \vec{0}} \frac{x^t A x}{x^t x}.$$

- (b) Let now $\lambda_1 > \lambda_2 > \dots > \lambda_n > 0$ (all eigenvalues are different and positive). Suppose that for some $a_m > 0$, we have

$$\frac{1}{a_m} A^m \longrightarrow B, \quad \text{as } m \rightarrow \infty.$$

Determine the rank of B , $\text{rank}(B) = ?$. Can you always find such a sequence $\{a_m\}$? Explain.

13. Let $B = (b_{ij})_{n \times n}$ be a real, non-singular matrix and let $\|\cdot\|$ be an arbitrary matrix norm in the space of all $n \times n$ real matrices $M_n(\mathbb{R})$.
- Show that there exists $\epsilon > 0$, such that the matrix $B + A$ is invertible, for any $A \in M_n(\mathbb{R})$ with $\|A\| < \epsilon$.
 - Suppose B is real and symmetric with eigenvalues λ_i , $1 \leq i \leq n$. Determine the radius of convergence of the series

$$\sum_{k=0}^{\infty} B^k z^k, \quad z \in \mathbb{R},$$

with respect to some (any) matrix norm.

14. Let $Y \sim \mathcal{N}(\vec{0}, \Sigma)$, $\Sigma = (\sigma_{ij})_{n \times n}$ be a multivariate normal random vector in \mathbb{R}^n with covariance matrix Σ and zero mean. Show that there exist n orthonormal *constant* vectors $\vec{x}_1, \dots, \vec{x}_n \in \mathbb{R}^n$, such that

$$Y \stackrel{d}{=} \sqrt{\lambda_1} Z_1 \vec{x}_1 + \dots + \sqrt{\lambda_n} Z_n \vec{x}_n,$$

where λ_j , $j = 1, \dots, n$ are the eigen values of Σ and where $Z_j, j = 1, \dots, n$ are independent standard Normal random variables.

15. (*Cochran's Theorem*) Suppose that $A_1 + \dots + A_k = I_n$, where A_i , $1 \leq i \leq k$ are real $n \times n$ symmetric matrices. Then show that the following statements are equivalent:

(i) $\sum_{i=1}^k \text{rank}(A_i) = n$

(ii) $A_i A_j = (0)$, for all $1 \leq i \neq j \leq k$

(iii) $A_i^2 = A_i$, for all $1 \leq i \leq k$.

16. Let $Z_j, j = 1, \dots, n$ be independent standard Normal random variables and consider the random vector $Z = (Z_j)_{j=1}^n$. Let $A + B = I_n$, where A and B are real symmetric matrices such that $\text{rank}(A) + \text{rank}(B) = n$. Using Cochran's theorem, show that:

(a) AZ and BZ are independent;

(b) Determine the distributions of $\|AZ\|^2$ and $\|BZ\|^2$ and show that they depend only on the ranks of the matrices A and B .