

## Assignment 2: Review of Real analysis and Classes of Sets

Due on Wed, Oct 21, 2009

- (functions and operations with sets) Let  $f : X \rightarrow Y$  and  $A_\alpha \subset X$ ,  $B_\alpha \subset Y$ , for  $\alpha \in \mathcal{A}$ . Prove rigorously the following properties:
  - $f(\cup_{\alpha \in \mathcal{A}} A_\alpha) = \cup_{\alpha \in \mathcal{A}} f(A_\alpha)$
  - $f(\cap_{\alpha \in \mathcal{A}} A_\alpha) \subset \cap_{\alpha \in \mathcal{A}} f(A_\alpha)$  and give an example of  $f$  where the inclusion is *strict*.
  - $f^{-1}(\cup_{\alpha \in \mathcal{A}} B_\alpha) = \cup_{\alpha \in \mathcal{A}} f^{-1}(B_\alpha)$
  - $f^{-1}(\cap_{\alpha \in \mathcal{A}} B_\alpha) = \cap_{\alpha \in \mathcal{A}} f^{-1}(B_\alpha)$
  - $f(C) \setminus f(A) \subset f(C \setminus A)$  and  $f^{-1}(D) \setminus f^{-1}(E) = f^{-1}(D \setminus E)$ , for all  $A, C \subset X$  and  $D, E \subset Y$ .
- Let  $(X, \rho)$  be a separable metric space and  $K \subset X$  be a compact set. Show that every closed set  $F \subset K$ , subset of  $K$ , is also compact.

- Thm (Bolzano–Weierstrass)* Let  $(X, \rho)$  be a complete metric space and  $K \subset X$  be a compact set. Show that any infinite sequence  $\{x_n\}_{n \in \mathbb{N}} \subset K$  has an infinite convergent sub-sequence  $\{x_{n_k}\}_{k \in \mathbb{N}}$ .
- Consider the set  $C[0, 1]$  of continuous, real-valued functions, defined on the interval  $[0, 1]$ . Equip  $C[0, 1]$  with the uniform convergence norm:

$$\|f\|_\infty := \sup_{x \in [0, 1]} |f(x)|.$$

Prove that  $C[0, 1]$  is a *complete* normed space.

- Consider a set of functions  $\{f_n\} \subset C[0, 1]$ . This set is said to be equi-continuous, if for all  $\epsilon > 0$ , and all  $x \in [0, 1]$  there exists  $\delta > 0$ , such that

$$|f_n(x) - f_n(y)| < \epsilon, \quad \text{for all } y \in [0, 1], |x - y| < \delta.$$

The functions are said to be uniformly bounded, if  $\sup_{n \in \mathbb{N}} \|f_n\|_\infty \leq C < \infty$ .

Prove the following result:

*Thm (Arzelà–Ascoli)* Let  $\{f_n\} \subset C[0, 1]$  be a set of equicontinuous and uniformly bounded functions. Then, there exists a subsequence  $\{f_{n_k}\}_{k \in \mathbb{N}}$  that converges uniformly (i.e. in the norm  $\|\cdot\|_\infty$ .)

- Let  $\mathcal{B}_\mathbb{R} = \sigma(\mathcal{C})$ , where  $\mathcal{C}$  is the class of all open sets in  $\mathbb{R}$  (in the usual topology, generated by the metric  $\rho(x, y) = |x - y|$ ,  $x, y \in \mathbb{R}$ .)
  - Show that for every open set  $U \subset \mathbb{R}$ , there exists a countable collection of open intervals  $(a_n, b_n)$ ,  $n \in \mathbb{N}$ , such that  $U = \cup_{n=1}^\infty (a_n, b_n)$ .
  - Using (a), show that  $\mathcal{B}_\mathbb{R} = \sigma(\mathcal{D})$ , where  $\mathcal{D} = \{[a, b], a, b \in \mathbb{Q}\}$ , and where  $\mathbb{Q}$  denotes the rational numbers.
- Give examples of:
  - a semiring that is not a ring

- b. a ring that is not an algebra
- c. an algebra that is not a  $\sigma$ -algebra
- d. a  $\sigma$ -algebra.

8. (*The Monotone Class Theorem*) A non-empty class  $\mathcal{M} \subset 2^X$  of subsets of  $X$ , is said to be a *monotone class* if: (i) For any  $A_n \in \mathcal{M}$ , such that  $A_n \uparrow A$ , it follows that  $A \in \mathcal{M}$ , i.e. if  $A_n \subset A_{n+1}$ ,  $n \in \mathbb{N}$ , we have  $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$ . and (ii) For any  $A_n \in \mathcal{M}$ , such that  $A_n \downarrow A$ , as  $n \rightarrow \infty$ , it follows that  $A \in \mathcal{M}$ .

That is, a class of sets is said to be monotone if it is closed with respect to monotone (increasing and decreasing) limits.

Prove that for any ring  $\mathcal{R} \subset 2^X$  on  $X$ , we have  $\mathcal{M}(\mathcal{R}) \equiv \mathcal{S}(\mathcal{R})$ , that is, the minimal monotone class containing  $\mathcal{R}$  is precisely the  $\sigma$ -ring generated by  $\mathcal{R}$ .

*Hint:* To prove ' $\supset$ ', show first that  $\mathcal{M}(\mathcal{R})$  is a ring and then prove that any ring which is also a monotone class is a  $\sigma$ -ring. To show that  $A \cup B \in \mathcal{M}(\mathcal{R})$ , for all  $A, B \in \mathcal{M}(\mathcal{R})$ , fix  $A \in \mathcal{R}$  and consider

$$\mathcal{C}_A := \{B \in \mathcal{M}(\mathcal{R}) : A \cup B \in \mathcal{M}(\mathcal{R})\}.$$

Prove the *Claim*:  $\mathcal{C}_A$  is a monotone class.

Use this fact to conclude that  $A \cup B \in \mathcal{M}(\mathcal{R})$ , for all  $A \in \mathcal{R}$  and  $B \in \mathcal{M}(\mathcal{R})$ . Repeat this construction with fixed  $B \in \mathcal{M}(\mathcal{R})$  and  $\mathcal{D}_B := \{A \in \mathcal{M}(\mathcal{R}) : A \cup B \in \mathcal{M}(\mathcal{R})\}$ , to show that  $A \cup B \in \mathcal{M}(\mathcal{R})$ , for all  $A \in \mathcal{R}$  and  $B \in \mathcal{M}(\mathcal{R})$ .

Provide a similar (but complete!) argument for  $A \setminus B$ .