

Assignment 2: Real Analysis & Classes of Sets

Due on Wednesday, October 15, 2008

Justify your answers to receive credit. No late homework will be accepted.

- Let $f : X \rightarrow Y$ and $A_\alpha \subset X$, $B_\alpha \subset Y$, for $\alpha \in \mathcal{A}$. Prove rigorously the following properties:
 - $f(\cup_{\alpha \in \mathcal{A}} A_\alpha) = \cup_{\alpha \in \mathcal{A}} f(A_\alpha)$
 - $f(\cap_{\alpha \in \mathcal{A}} A_\alpha) \subset \cap_{\alpha \in \mathcal{A}} f(A_\alpha)$ and give an example of f where the inclusion is *strict*.
 - $f^{-1}(\cup_{\alpha \in \mathcal{A}} B_\alpha) = \cup_{\alpha \in \mathcal{A}} f^{-1}(B_\alpha)$
 - $f^{-1}(\cap_{\alpha \in \mathcal{A}} B_\alpha) = \cap_{\alpha \in \mathcal{A}} f^{-1}(B_\alpha)$
 - $f(C) \setminus f(A) \subset f(C \setminus A)$ and $f^{-1}(D) \setminus f^{-1}(E) = f^{-1}(D \setminus E)$, for all $A, C \subset X$ and $D, E \subset Y$.
- Prove that \mathbb{R}^d with the usual Euclidean metric is a separable metric space.
 - Prove that for any open set $U \subset \mathbb{R}^d$, there exist $x_n \in \mathbb{R}^d$ and $\epsilon_n > 0$, such that $U = \cup_{n \in \mathbb{N}} B(x_n, \epsilon_n)$.

Hint: For **b.** let $A \subset U$ be a countable dense subset of U . For every $x \in A$, let $\epsilon_x := \sup\{\epsilon > 0 : B(x, \epsilon) \subset U\}$. Show first that $B(x, \epsilon_x) \subset U$ and then use the special construction of this ball to show that $U = \cup_{x \in A} B(x, \epsilon_x)$.
- Prove the Bolzano–Weierstrass theorem in \mathbb{R} , that is, show that every closed and bounded sequence has a convergent sub–sequence (a limit point).
 - Let (X, ρ) be a complete metric space. Show that if $\{x_n\}_{n \in \mathbb{N}} \subset K$, for some compact set $K \subset X$, then show that $\{x_n\}$ has a convergent subsequence.
- Let $f : X_1 \rightarrow X_2$ be a continuous function between the metric spaces (X_1, ρ_1) and (X_2, ρ_2) . Suppose that $K \subset X_1$ is compact.
 - Show that $f(K)$ is also compact subset of X_2 .
 - Show that f is uniformly continuous on K .

Hint: For **b.** by continuity, we have that $\forall \epsilon > 0$ and $\forall y \in f(K)$, $y = f(x)$, there exists $\delta = \delta(x) > 0$, such that $\rho_2(f(x'), f(x)) < \epsilon/2$, for all $x' \in B(x, \delta)$. Consider the open cover $B(x, \delta(x)/2)$, $x \in K$ of the compact K .
- Consider the set of *bounded*¹ continuous functions $C_b(X_1, X_2) = \{f : X_1 \rightarrow X_2\}$ between the metric spaces (X_1, ρ_1) and (X_2, ρ_2) .

Consider the functional

$$d(f, g) := \sup_{x \in X_1} \rho_2(f(x), g(x)).$$

- Prove that d is a metric on $C_b(X_1, X_2)$. The metric d is called the uniform metric.
- If X_2 is *complete*, then show that the metric space $(C_b(X_1, X_2), d)$ is *complete*.

¹Here $f : X_1 \rightarrow X_2$ is bounded if $f(X_1)$ is contained in a ball of finite radius.

Hint: For **(b)**, consider a Cauchy sequence $\{f_n\}_{n \in \mathbb{N}} \subset C(X_1, X_2)$ in the uniform metric d .
(i) Select an infinite subsequence $\{n_k\} \subset \mathbb{N}$, such that $d(f_{n_k}, f_{n_{k+1}}) \leq 2^{-k}$, for all $k \in \mathbb{N}$. *(ii)* Using this sequence, construct a function $f : X_1 \rightarrow X_2$, such that $\rho_2(f_{n_k}(x), f(x)) \rightarrow 0$, as $k \rightarrow \infty$, for all $x \in X_1$, that is, $f(\cdot)$ is the point-wise limit of the functions $f_{n_k}(\cdot)$. *(iii)* Prove that $\sup_{x \in X_1} \rho_2(f(x), f_{n_k}(x)) \rightarrow 0$, as $k \rightarrow \infty$ and that f is continuous.

6. Consider two complete separable metric spaces (X_i, ρ_i) , $i = 1, 2$. For a sequence of functions $f_n : X_1 \rightarrow X_2$, for $n \in \mathbb{N}$, we say that $f_n \rightarrow f$ *continuously*, if $\rho_2(f_n(x_n), f(x)) \rightarrow 0$, as $n \rightarrow \infty$, for any $x_n \in X_1$, such that $\rho_1(x_n, x) \rightarrow 0$, as $n \rightarrow \infty$.

Let $f_n : X_1 \rightarrow X_2$ be continuous and suppose that X_1 is *compact*. Prove that $f_n \rightarrow f$ continuously to a *continuous* function $f : X_1 \rightarrow X_2$, if and only if f_n converges uniformly to f .

7. *More to be added...*